Unit III

3 Stability Analysis

3.1 : Characteristic Equation, Poles and Zeros

Important Points to Remember

- The closed loop transfer function of a system is given by, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$
- The equation obtained by equating denominator of a closed loop transfer function to zero is called **characteristic equation** of a system. It is given by, $1+G(s)H(s) = 0$.
- The roots of the characteristic equation of a system are called closed loop poles of that system.
- The roots of a equation obtained by equating numerator of a closed loop transfer function to zero are called **zeros** of that system.
- The system stability depends on the locations of closed loop poles of a system in s-plane hence characteristic equation giving closed loop poles of a system plays an important role in the stability analysis of a system.

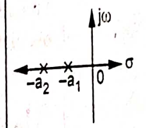
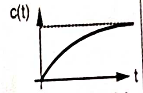
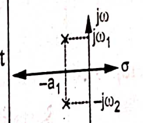
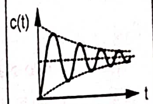
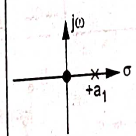
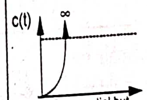
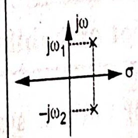
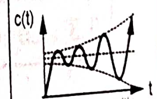
3.2 : Response of Various Pole Locations in s-plane

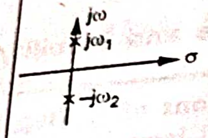
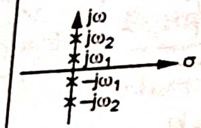
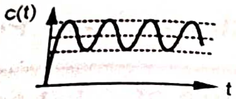
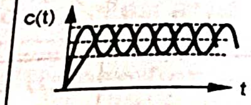
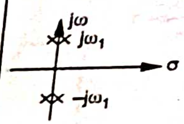
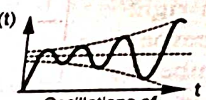
Q.1 Show the various pole locations in s-plane and the corresponding response of a system. Comment on the stability.

☞[SPPU : May-13, Marks 6]

Ans. :

Loacation of Closed Loop Poles and Stability Condition

Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards ∞	Unstable
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		 Oscillations with increasing amplitude	Unstable

5.	Non repeated pair on imaginary axis without any pole in R.H.S. of s-plane	 or  two non repeated pairs on imaginary axis.	 Frequency of oscillations = ω_1  Sustained oscillations with two frequency components ω_1 and ω_2	Marginally or critically stable Marginally or critically stable.
6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane		 Oscillations of increasing amplitude	Unstable

3.3 : Concept of Stability

Q.2 Define : i) Stable system ii) Unstable system ii) Critically or Marginally stable system iv) Conditionally stable system

☞[SPPU : Dec.-99, 01, 02, 05, May-01, 03, 04, 07, Marks 7]

Ans. : i) A linear time invariant system is said to be **stable** if following conditions are satisfied :

a) When the system is excited by a bounded input, output is also bounded and controllable.

b) In the absence of the input, output must tend to zero irrespective of the initial conditions.

- This is called bounded input bounded output (BIBO) stability.

ii) A linear time invariant system is said to be **unstable** if,

a) For a bounded input, it produces an unbounded output.

b) In absence of the input, output may not return to zero. It shows certain output without input.

iii) A linear time invariant system is said to be **critically or marginally stable** if for a bounded input, its output oscillates with constant frequency and amplitude. Such oscillations of output are called **undamped oscillations or sustained oscillations.**

iv) A linear time invariant system is said to be **conditionally stable** if for a certain condition of a particular parameter of the system, its output is bounded one. Otherwise if that condition is violated output becomes unbounded and system becomes unstable. Thus the stability of the system depends on condition of a particular parameter of the system. Such a system is called **conditionally stable system.**

3.4 : Relative Stability

Q.3 Explain the concept of relative stability in brief.

Ans. :

- The stability of a particular system defined based on the locations of closed loop poles in s-plane is called its abolute stability. While the relative stability of a system is always defined by comparing its settling time with other system.
- System is said to be relatively more stable if settling time for that system is less than that of the other system.
- The settling time of the root or pair of complex conjugate roots of the characteristic equation i.e. closed loop poles, is inversely proportional to the real part of the roots.
- So for the roots located near the $j\omega$ axis, settling time will be large. As roots or pair of complex conjugate roots moves away from $j\omega$ axis i.e. towards left half of s-plane, settling time becomes lesser or smaller and system becomes more and more stable.

- This is because as the closed loop poles move away from the imaginary axis in left half of s-plane, the transient part of the output dies out more and more quickly and the system settles to a steady state value very quickly.
- So relative stability of the system improves, as the closed loop poles move away from the imaginary axis in left half of s-plane.

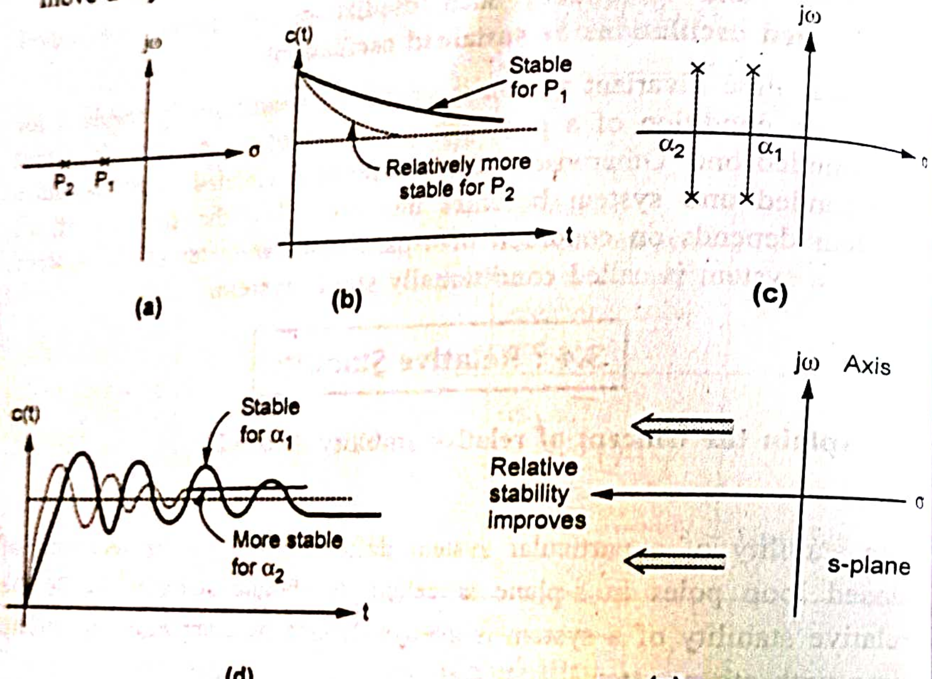


Fig. Q.3.1 Concept of relative stability

- The Fig. Q.3.1 shows the relative stability related to real and pair of complex conjugate closed loop poles.
- Hence the roots of characteristic equation which are located near the imaginary axis of s-plane are called **dominant roots** which decide the stabiliy of the system.

3.5 : Routh-Hurwitz Stability Criterion

Important Points to Remember

- In order that the characteristic equation of a system has no root in right of s-plane, it is **necessary but not sufficient** that,
  1) All the coefficients of the polynomial have the same sign.
  2) None of the coefficient vanishes i.e. all powers of 's' must be present in descending order from 'n' to zero.

Q.4 State and explain Routh's stability criterion.

[SPPU : Dec.-01, 07, May-2000, 03, 05, 09, Marks 6]

Ans. : Consider the general characteristic equation as,

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots\ldots + a_n = 0$$

- The Routh-Hurwitz array is then obtained as,

s^n	a_0	a_2	a_4	a_6	....
s^{n-1}	a_1	a_3	a_5	a_7	
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
:	:	:	:		
s^0	a_n				

- Coefficients for first two rows are written directly from characteristic equation.

From these two rows, next rows can be obtained as follows.

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

- From 2nd and 3rd row, 4th row can be obtained as

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

- This process is to be continued till the coefficient for s^0 is obtained which will be a_n. From this array stability of a system can be predicted.

- The necessary and sufficient condition for system to be stable is,

"All the terms in the first column of Routh's array must have same sign. There should not be any sign change in the first column of Routh's array."

- If there are any sign changes existing then,

a) System is unstable.

b) The number of sign changes equals the number of roots lying in the right half of the s-plane.

Q.5 Investigate the stability of a system having closed loop characteristic equation : $Q(s) = s^4 + 5s^3 + 7s^2 + 3s + 2$ using Routh stability criterion. Also find number of closed loop poles in the right half of s-plane. **[SPPU : Dec.-17, Marks 4]**

Ans. : Routh's array is,

s^4	1	7	2
s^3	5	3	0
s^2	6.4	2	0
s^1	1.4375	0	
s^0	2		

No sign change in the first column hence no closed loop pole in the right half of s plane, system is stable.

Q.6 Investigate the stability of system with characteristic equation $Q(s) = s^4 + 6s^3 + 15s^2 + 5s + 3 = 0$ **[SPPU : May-18, Marks 4]**

Ans. : Routh's array is,

s^4	1	15	3
s^3	6	5	0
s^2	14.1667	3	
s^1	3.73	0	
s^0	3		

There are no sign changes in the first column of array hence system is stable in nature.

Q.7 Investigate the stability of system with characteristic equation : $Q(s) = s^4 + 9s^3 + 7s^2 + 4s + 3 = 0$ using Routh stability test. Also determine the number of poles in the right half of s-plane.

[SPPU : Dec.-18, Marks 4]

Ans. : Routh's array is,

s^4	1	7	3
s^3	9	4	0
s^2	6.555	3	0
s^1	− 0.119	0	
s^0	3		

There are two sign changes in the first column of the array.

So there are 2 poles in the right half of s-plane. System is unstable.

Q.8 : Investigate the stability of the system with characteristic equation : $s^4 + 2s^3 + 4s^2 + 6s + 8 = 0$. How many poles of system lie in right half of s-plane ? **[SPPU : Dec.-19, Marks 4]**

Ans. : The Routh's array is,

s^4	1	4	8
s^3	2	6	0
s^2	1	8	0
s^1	−10	0	
s^0	8		

There are two sign changes in the first column of the Routh's array hence two poles of system lie in right half of s-plane.

3.6 : Special Cases of Routh's Array

Q.9 Discuss the special cases of Routh's array.

[SPPU : May-02, 07, Dec.-04, Marks 6]

Ans. : There are two special cases of Routh's array.

Special case 1 : The appearance of first element of any row as zero while same remaining row consisting of at least one nonzero element.

- In such a case, that zero is replaced by a small positive number $+\varepsilon$ and array is completed interms of ε.
- The signs of the first column elements are examined by taking $\lim \varepsilon \to 0$ and the stability is predicted.

Special case 2 : There occurs a complete row as row of zeros while generating an array. So array terminates abruptly as shown,

s^n	a_1	a_2	a_3	
s^{n-1}	b_1	b_2	b_3	
s^{n-2}	c_1	c_2	c_3	
s^{n-3}	0	0	0	⇐ Row of zeros

- In such case an equation is formed using the coefficients of a row which is just above the row of zeros. Such an equation is called an **Auxiliary Equation,** denoted as $A(s)$. The auxiliary equation is always odd or even polynomial in s.

$$A(s) = c_1 \ s^{n-2} + c_2 \ s^{n-4} + c_3 \ s^{n-6} \ \ldots\ldots$$

- Differentiate this with respect to s.

$$\frac{dA(s)}{ds} = c_1 (n-2) \ s^{n-3} + c_2 (n-4) \ s^{n-5} + c_3 (n-6) \ s^{n-7} \ \ldots\ldots$$

- Complete the array, replacing row of zeros by the coefficients of the equation $dA(s)/ds$.
- If at all, there is any sign change in the first column of the completed array then the given system is unstable.
- But if there is no sign change in the first column then it is confirmed that there is no closed loop pole in right half of s-plane. But system may be stable and its stability is determined by solving the equation $A(s) = 0$ i.e. auxiliary equation for its roots. From the locations of these roots it must be decided whether system is stable or not.

> Remember that the roots of $A(s) = 0$ are the **dominant roots** of $F(s) = 0$ and stability is totally dependent on the locations of roots of $A(s) = 0$ in such special case.

Q.10 Comment on stability using Routh criteria if characteristic equation is : $Q(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 5s + 6 = 0$. How many poles lie in right half of s-plane ? ☞**[SPPU : Dec.-14, Marks 4]**

Ans. : The Routh's array is,

s^5	1	3	5	
s^4	2	4	6	
s^3	1	2	0	
s^2	0 / $+\varepsilon$	6		⇐ Special case 1; $+\varepsilon$ is +ve
s^1	$\frac{2\varepsilon - 6}{\varepsilon}$	0		$\lim_{\varepsilon \to 0} \frac{2\varepsilon - 6}{\varepsilon} \to -\text{Ve}$
s^0	6			

> Thus there are two sign changes in the first column hence two poles lie in right half of s-plane.

Q.11 Investigate the stability of the system with characteristic equation : $Q(s) = s^4 + 6s^3 + 11s^2 + 6s + 10 = 0$. Comment on stability. ☞**[SPPU : May-19, Dec.-22, Marks 4]**

Ans. : The Routh's array is,

s^4	1	11	10	
s^3	6	6	0	
s^2	10	10	0	
s^1	0	0		⇐ Special case 2; $A(s) = 10s^2 + 10$
	20	0		$\therefore \frac{dA(s)}{ds} = 20s$
s^0	10			

No sign change in the first column so system may be stable. But due to special case 2, solve A(s) = 0

$\therefore \quad 10s^2 + 10 = 0 \quad$ i.e. $\quad s^2 = -1 \quad$ i.e. $\quad s = \pm j$

Thus there are nonrepeated roots purely on the imaginary axis hence system is **marginally stable.**

Q.12 Construct Routh array and determine the stability of the system whose characteristic equation is $s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$.

[SPPU : May-18, Marks 7, May-22, Dec.-22, Marks 8]

Ans. : The Routh's array is,

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0
	8	24	0	
s^2	6	16	0	
s^1	2.667	0		
s^0	16			

$\Rightarrow$ Special case 2

$A(s) = 2s^4 + 12s^2 + 16$

$\therefore \frac{dA(s)}{ds} = 8s^3 + 24s$

There is no sign change in the first column of completed array. But due to special case 2, **system may be stable.**

To obtain stability, solve $A(s) = 0$

i.e. $\quad 2s^4 + 12s^2 + 16 = 0$

$\therefore \quad s^2 = -2, -4$

i.e. $\quad s = \pm j\sqrt{2}, \pm j2$

As the roots of A(s) = 0 are complex conjugates, purely imaginary and non-repeated, the system is **marginally stable in nature.**

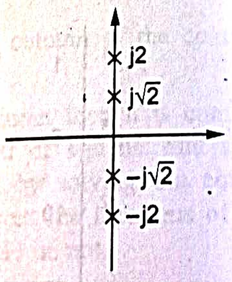


Fig. Q.12.1

3.7 : Marginal K and Frequency of Sustained Oscillations

Important Points to Remember

- Practically gain K of the forward path is unknown. The range of K for stability can be obtained by constructing Routh's array interms of K and then finding the range of K which will not create any sign change in the first column.
- The value of K which makes the system marginally stable is called marginal value of K denoted as K_{mar}. This is the value of K which makes any row other than s^0 as row of zeros in the Routh's array.
- To obtain the frequency of oscillations, solve the auxiliary equation A(s) = 0 for $K = K_{mar}$.
- The magnitude of imaginary roots of A(s) = 0 obtained for marginal value of K (K_{mar}) indicates the frequency of sustained oscillations, which system will produce under marginally stable condition.

Q.13 Investigate the stability of a system having closed loop characteristic equation : $Q(s) = s^3 + 7s^2 + 10s + K = 0$ and find K_{mar} and ω_{mar}.

[SPPU : May-17, Marks 4]

Ans. : Routh's array is,

s^3	1	10	From s^0 row, K > 0
s^2	7	K	From s^1 row, 70 − K > 0
s^1	$\frac{70-K}{7}$	0	$\therefore$ K < 70
s^0	K		

$\therefore$ $\boxed{0 < K < 70}$ is the range of K for stability.

The value of making row of s^1 as row of zeros is K_{mar} i.e., $K_{mar} = 70$

$A(s) = 7s^2 + K = 0 \quad$ i.e. $\quad 7s^2 = -K_{mar}$

$$\therefore \quad s^2 = \frac{-70}{7} = -10 \quad \text{i.e} \quad s = \pm j\, 3.1622$$

$$\therefore \quad \omega_{mar} = 3.1622 \text{ rad/sec}$$

3.8 : Relative Stability using Routh's-Hurwitz Criterion

Important Points to Remember

- If it is required to find relative stability of system about a line $s = -\sigma$, then substitute $s = s' - \sigma$, (σ = Constant) in characteristic equation and complete the array interms of s'. The number of sign changes in first column is equal to number of roots those are located to right of the vertical line $s = -\sigma$.

Q.14 The characteristics equation of closed loop system is given as $1 + G(s)\ H(s) = s^3 + 7s^2 + 25s + 39 = 0$. Determine the number of roots which are lying on left half side of $\sigma = -1$.

[SPPU : May-22, Marks 8]

Ans. : Use $s = s' - 1$ in the equation.

$$\therefore \quad (s'-1)^3 + 7(s'-1)^2 + 25(s'-1) + 39 = 0$$

$$\therefore \quad (s')^3 + 4(s')^2 + 14(s') + 20 = 0$$

Routh's array is,

$(s')^3$	1	14	No sign change in the first column hence no root is located to the right of $s = -1$.
$(s')^2$	4	20	
$(s')^1$	9	0	
$(s')^0$	20		

Hence number of roots lying to the left half of $s = -1$ is 3.

3.9 : Root Locus : Definition, Angle and Magnitude Conditions

Q.15 Define root locus. State the angle and magnitude condition of root locus. Mention the use of these conditions.

Ans. :

- The locus of the closed loop poles of a system, obtained when system gain 'K' is varied from 0 to $+\infty$ is called **Root Locus.**

- Angle condition can be stated as, $\angle G(s)H(s)$ for any value of 's' which is the root of equation $[1 + G(s)H(s) = 0]$ is.
  $= \pm (2q+1)\ 180°$ where $q = 0, 1, 2$ ....
  i.e. Odd multiple of 180°.

- Any point in s-plane which satisfies the angle condition has to be on the root locus of the corresponding system.

- Magnitude condition is stated as,

$$\left| G(s)H(s) \right|_{\substack{\text{at a point in s - plane} \\ \text{which is on root locus}}} = 1$$

- Magnitude condition is used to find K for any point on the root locus. But the magnitude condition can be used only when it is sure that a point is on the root locus.
- Hence angle condition is used first to confirm whether a point is on the root locus or not and then once confirmed it is followed by the magnitude condition to find the corresponding calue of K.

3.10 : Rules to Construct Root Locus

Q.16 Explain the rules to construct the root locus.

[SPPU : May-02, 04, 05, Dec.-01, 06, 08, Marks 8]

Ans. : • The various rules to construct the root locus are,

Rule 1 : The root locus is always symmetrical about the real axis.

Rule 2 : If $G(s)H(s)$ = Open loop T.F. of the system and

P = Number of open loop poles, Z = Number of open loop zeros then Number of branches equal to number of open loop poles (N = P).

- Branches will start from each of the location of open loop pole. Out of 'P' number of branches, 'Z' number of branches will terminate at the locations of open loop zeros. The remaining 'P – Z' branches will approach to infinity. The branch direction always remains from open loop poles towards open loop zeros.

Rule 3 : A point on the real axis lies on the root locus if the sum of the number of open loop poles and the open loop zeros, on the real axis, to the right hand side of this point is odd.

Rule 4 : Angles of aysmptotes

- The 'P–Z' branches approach to infinity along the straight lines called **Asymptotes** of the root locus. Asymptotes are the guidelines for the branches approaching to infinity.
- Angles of such asymptotes are given by ,

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \qquad \text{where } q = 0, 1, 2 \ldots\ (P-Z-1)$$

Rule 5 : Centroid

- All the asymptotes intersect the real axis at a common point known as **centroid** denoted by σ. The co-ordinates of centroid can be calculated as,

$$\sigma\ (\text{centroid}) = \frac{\sum \text{Real parts of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P-Z}$$

- Centroid is always real, it may be located on negative or positive real axis. It may or may not be the part of the root locus.

Rule 6 : Breakaway point

- Breakaway point is a point on the root locus where multiple roots of the characteristic equation occurs, for a particular value of K.

- Steps to determine the co-ordinates of breakaway points are,

Step 1 : Construct the characteristic equation 1 + G(s)H(s) = 0 of the system.

Step 2 : From this equation, separate the terms involving 'K' and terms involving 's'. Write the value of K in terms of s i.e. K = F(s).

Step 3 : Differentiate above equation w.r.t. 's', equate it to zero

i.e. $\frac{dK}{ds} = 0$

Step 4 : Roots of the equation $\frac{dK}{ds} = 0$ gives us the breakaway points.

- Out of all the roots of $\frac{dK}{ds} = 0$, those for which the value of K is positive are valid for the root locus.
- The root locus branches always leave breakaway points at an angle of $\pm\frac{180^\circ}{n}$ where n = Number of branches approaching at breakaway point.

Rule No. 7 : Intersection of root locus with imaginary axis

- This can be determined by constructing Routh's array interms of K. Find the marginal value of K. Solve A(s) = 0 for marginal value of K. The roots of A(s) = 0 for K_{mar} are the intersection points of root locus with the imaginary axis.

Rule No. 8 : Angle of departure at complex conjugate poles and angle of arrival at complex conjugate zeros.

- The angle of departure is the angle at which a branch departs from a complex pole denotes as ϕ_d. It is obtained as,

$$\phi_d = 180^\circ - \phi \qquad \text{where } \phi = \sum \phi_P - \sum \phi_Z$$

where $\sum \phi_P$ = Contributions by the angles made by remaining open loop poles at the pole at which ϕ_d is to be calculated.

$\sum \phi_Z$ = Contributions by the angles made by the open loop zeros at the pole at which ϕ_d is to be calculated.

- The angle of arrival is the angle at which a branch arrives at a complex zero, denotes as ϕ_a.
- Angle of arrival at a complex zero is given by,

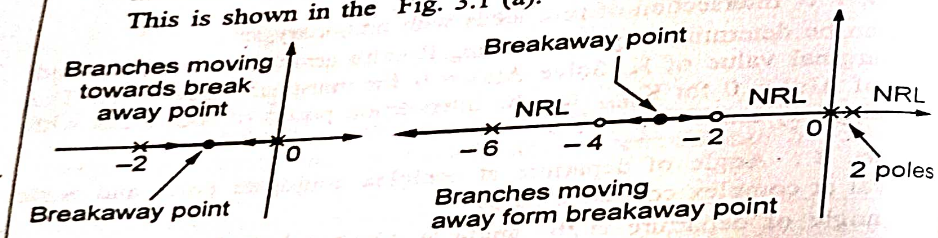
$$\phi_a = 180° + \phi \quad \text{where } \phi = \sum \phi_P - \sum \phi_Z$$

- To calculate $\Sigma\, \phi_P$, join all the remaining poles to the complex pole under consideration. Add all the angles subtended by these phasors with respect to positive x-axis. Similarly join all the zeros to pole under consideration and adding all angles subtended by these phasors with respect to positive x-axis, determine $\Sigma\, \phi_Z$.

Important Points to Remember

General predictions for existence of breakaway point in root locus :

1) If there are adjacently placed poles on the real axis and the real axis between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed poles. This is shown in the Fig. 3.1 (a).



(a) Fig. 3.1 (b)

2) If there are two adjacently placed zeros on real axis and section of real axis in between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed zeros. This is shown in the Fig. 3.1 (b).

3) If there is a zero on the real axis and to the left of that zero there is no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one breakaway point to the left of that zero. This is shown in the Fig. 3.1 (c).

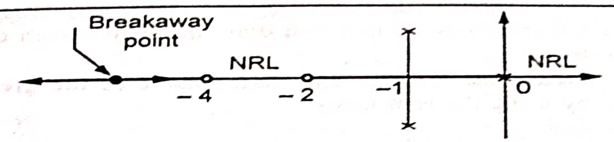


Fig. 3.1 (c)

3.11 : Steps to Draw Root Locus

Q.17 List the steps to draw the root locus for a given system.

Ans. :

Step 1 : Get the general information about number of open loop poles, zeros, number of branches etc. from G(s)H(s).

Step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions.

Step 3 : Calculate angles of asymptotes,

$$\theta = \frac{(2\theta + 1)\, 180°}{P - Z} \quad \text{where } \theta = 0, 1, 2 \ldots. (P - Z - 1)$$

Step 4 : Determine the centroid.

$$\sigma(\text{centroid}) = \frac{\sum \text{Real parts of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P - Z}$$

Step 5 : Calculate the breakaway and breakin points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as breakaway points.

Step 6 : Calculate the intersection points of root locus with the imaginary axis. These are the roots of the equation A(s) = 0 for marginal value of K, obtained from Routh's array.

Step 7 : Calculate the angles of departures or arrivals for the complex conjugate poles and zeros, if present.

Step 8 : Combine steps 1 to 7 and draw the final sketch of the root locus.

Step 9 : Predict the stability and performance of the given system by using the root locus.

Q.18 Open loop transfer function of unity feedback system is $G(s) = \frac{K}{s(s+2)(s+10)}$. Sketch the complete root locus and comment on stability of system. **[SPPU : Dec.-15, Marks 8]**

Ans. :

Step 1 : $P = 3$, $Z = 0$, $N = 3$,
$P - Z = 3$ branches to ∞
Starting : $s = 0, -2, -10$
Terminating $= \infty, \infty, \infty$

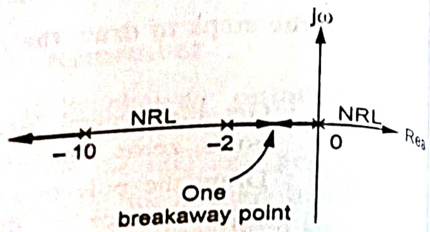


Fig. Q.18.1

Step 2 : Sections of real axis as shown in the Fig. Q.18.1.

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)180°}{P-Z}, \quad q = 0, 1, 2$$

$$\therefore \quad \theta_1 = 60°, \quad \theta_2 = 180°, \quad \theta_3 = 300°$$

Step 4 : Centroid $= \dfrac{\sum \text{R.P. of O.L. poles} - \sum \text{R.P. of O.L. zeros}}{P-Z}$

$$= \frac{0-2-10}{3} = 2 - 4$$

Step 5 : Breakaway point, $1 + G(s)H(s) = 0$

$$\therefore \quad 1+\frac{K}{s(s+2)(s+10)} = 0 \text{ i.e. } s^3+12s^2+20s+K = 0 \quad \ldots (Q.18.1)$$

$$\therefore \quad K = -s^3-12s^2-20s \quad \ldots (Q.18.2)$$

$$\therefore \quad \frac{dK}{ds} = 0 \text{ gives } 3s^2+24s+20 = 0 \text{ i.e. } s = -0.95, -7.05$$

$\therefore$ $s = -0.95$ is valid breakaway point with $K = +5.59$ from equation (Q.18.2).

Step 6 : Intersection with jω axis : From equation (Q.18.1),

s^3	1	20
s^2	12	K
s^1	$\frac{240-K}{12}$	0
s^0	K	

$\therefore \quad 240-K = 0$ for $K = K_{mar}$

$\therefore \quad K_{mar} = 240$

$12s^2 + K_{mar} = 0$ is $A(s) = 0$

$\therefore \quad s^2 = -20$ i.e. $s = \pm j4.47$

Step 7 : No complex poles hence angles of departure not required.

Step 8 : The complete root locus is shown in the Fig. Q.18.2.

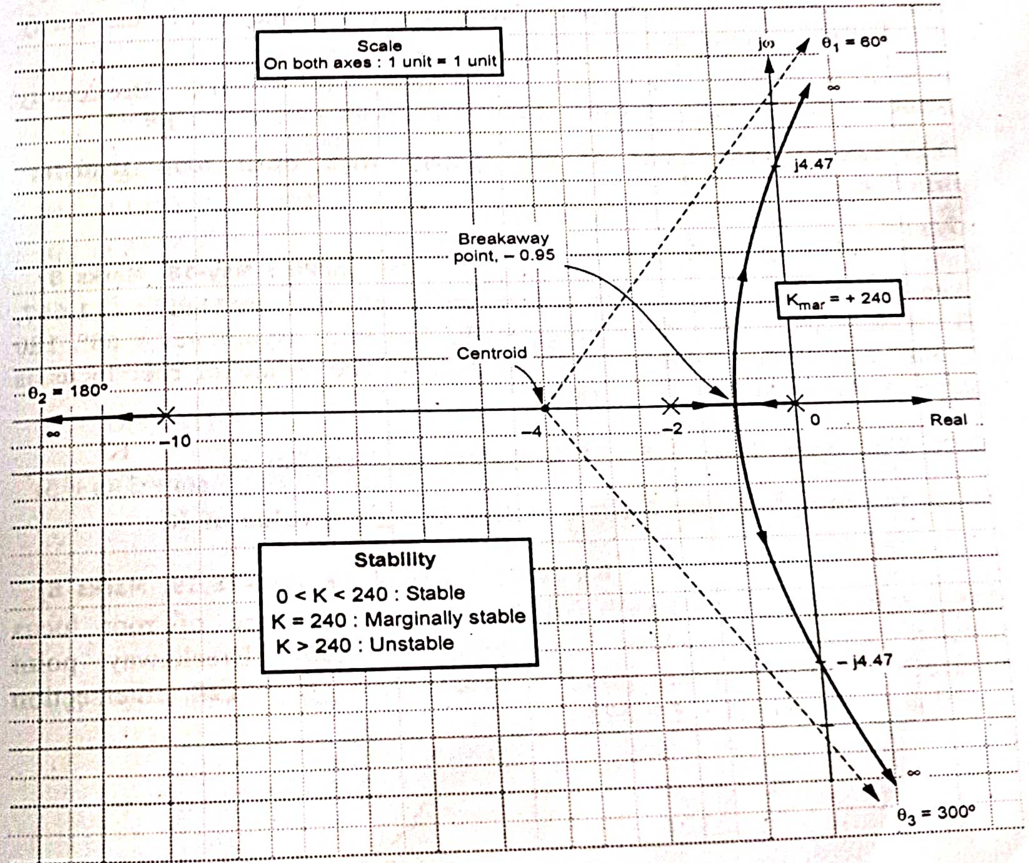


Fig. Q.18.2

Q.19 Sketch root locus of a system with open loop transfer function

$G(s)\ H(s) = \dfrac{K}{s(s+4)(s+6)}$ **[SPPU : May-17, Marks 8]**

Ans. : Refer Q.18 for the procedure and nature of the root locus. Verify the important points as : Centroid = **– 3.333**, θ_1 = **60°**, θ_2 = **180°**, θ_3 = **300°**, Breakaway point = **– 1.57**, K_{mar} = **240**, Intersection with jω axis = **± j 4.898**.

Q.20 Sketch root locus of a system with open loop transfer function

$G(s)\ H(s) = \dfrac{K}{s(s+2)(s+8)}$. **[SPPU : Dec.-17, Marks 8]**

Ans. : Refer Q.18 for the procedure and nature of the root locus. Verify the important points as :

centroid = **– 3.333**, θ_1 = **60°**, θ_2 = **180°**, θ_3 = **300°**, Breakaway point = **– 0.93**, K_{mar} = **160**, Intersection with jω axis = **± j 4**.

Q.21 Sketch root locus of the system with open loop transfer function :

$G(s) = \dfrac{K}{s(s+2)(s+3)}$ **[SPPU : May-18, Marks 8]**

Ans. : Refer Q.18 for the procedure and verify that centroid = **– 1.667**, Breakaway point **s = – 0.784**, K_{mar} = **30**, Angles of asymptotes = **60°, 180°, 300°** Intersection with jω axis = **± j 2.45**. The nature of root locus is same as shown in the Fig. Q.18.2.

Q.22 The open loop T.F. of a system is $G(s)H(s) = \dfrac{K}{s(s+3)(s+5)}$.

Draw the complete root locus and find marginal value of K.

[SPPU : May-14,19, Dec.-18,19, Marks 8]

Ans. : Refer the procedure used in Q.18. The nature of root locus remains same. Verify that : Centroid = **–2.667**, Breakaway point = **–1.213**, K at breakaway point = **+10.2837**, K_{mar} = **120**, Intersection with imaginary axis at **± j 3.873**.

Q.23 A unity feedback system has the loop transfer function.

$G(s) = \dfrac{K}{s(s+1)(s+3)(s+4)}$ **Plot root loci by determining breakaway points and intersection with imaginary axis.**

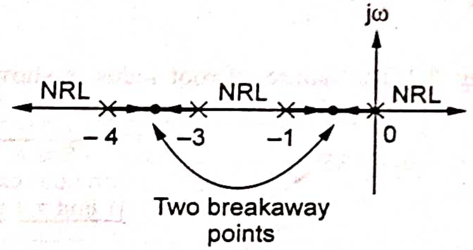
[SPPU : Dec.-12,22, May-16, Marks 12]

Ans. : P – Z = 4 branches to ∞

Starting points : s = 0, – 1, – 3, – 4

Terminating points : ∞, ∞, ∞, ∞

Step 2 : Sections of real axis. Two breakaway points possible



Step 3 : Angles of asymptotes,

$\theta_1 = 45°$,
$\theta_2 = 135°$, $\theta_3 = 225°$,
$\theta_4 = 315°$

Step 4 : Centroid = $\dfrac{0-1-3-4}{4} = -\,\mathbf{2}$

Step 5 : Breakaway points

$1 + G(s)H(s) = 0$

i.e. $s^4 + 8s^3 + 19s^2 + 12s + K = 0$

$\therefore \quad K = -s^4 - 8s^3 - 19s^2 - 12s$

$\therefore \quad \dfrac{dK}{ds} = -4s^3 - 24s^2 - 38s - 12 = 0$

i.e. $s^3 + 6s^2 + 9.5\,s + 3 = 0$

$\therefore \quad s = \mathbf{-0.4188,\ -3.5811}$ ... Breakaway points

Step 6 : Intersection with jω axis.

s^4	1	19	K
s^3	8	12	0
s^2	17.5	K	
s^1	$\frac{210-8K}{17.5}$	0	
s^0	K		

$\therefore\ 210 - 8K = 0$

$\therefore\ K_{mar} = \frac{210}{8} = 26.25$

$A(s) = 17.5\, s^2 + K_{mar} = 0$

$\therefore\ s^2 = -\frac{26.25}{17.5}$ i.e. $s = \pm j\, 1.224$

Step 7 : The nature of root locus is shown in the Fig. Q.23.1.

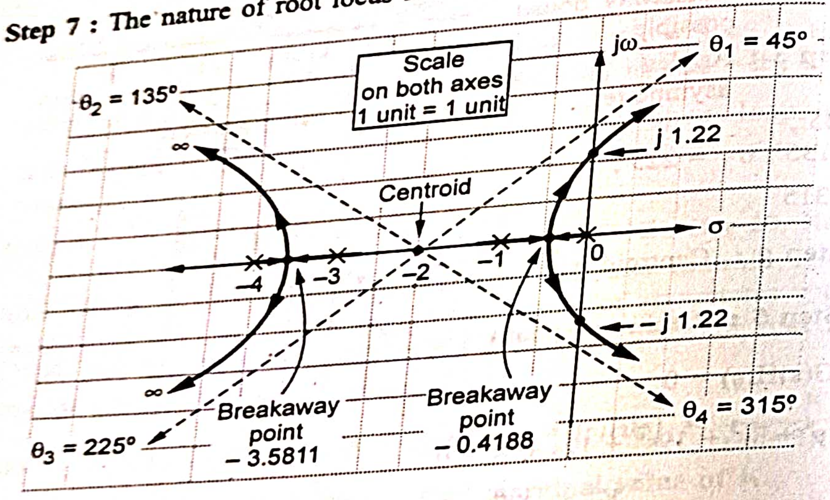


Fig. Q.23.1

Q.24 A unity feedback system with open loop transfer function $G(s) = \frac{K}{(s+1)^4}$. Plot root locus.

[SPPU : May-22, Marks 10]

Ans. : Step 1 : $P = 4,\ Z = 0,\ N = 4,\ P - Z = 4 \rightarrow \infty$

Starting : $s = -1, -1, -1, -1$

Terminating : $\infty, \infty, \infty, \infty$

Step 2 : Sections of real axis as shown in the Fig. Q.24.1(a).

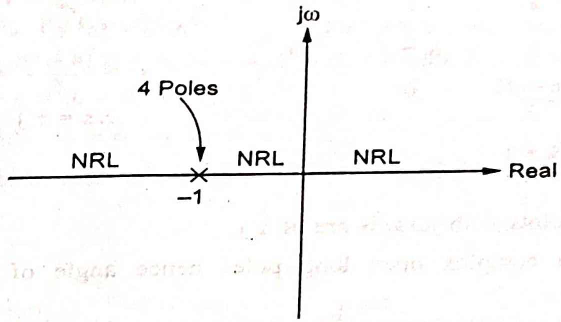


Fig. Q.24.1 (a)

Step 3 : $\theta = \frac{(2q+1)180°}{P-Z}$, $q = 0, 1, 2, 3$

$\therefore \theta_1 = 45°,\ \theta_2 = 135°,\ \theta_3 = 225°,\ \theta_4 = 315°$

Step 4 : Centroid $= \frac{-1-1-1-1-0}{4} = -1$

Step 5 : Breakaway point

$$1 + G(s)H(s) = 0 \text{ gives } 1 + \frac{K}{(s+1)^4} = 0$$

$$\therefore\ K = -s^4 - 4s^3 - 6s^2 - 4s - 1 \quad ...(Q.24.1)$$

$$\therefore\ \frac{dK}{ds} = 0 \text{ gives } 4s^3 + 12s^2 + 12s + 4 = 0$$

Solving, $s = -1, -1, -1$

At $s = -1$, $K = 0$ from equation (Q.24.1) hence all branches meet together at $s = -1$ at start and then approach to ∞ along asymptotes.

Step 6 : Intersection with jω axis

$1 + G(s)\, H(s) = 0$ gives $s^4 + 4s^3 + 6s^2 + 4s + 1 + K = 0$

s^4	1	6	K + 1
s^3	4	4	0
s^2	5	K + 1	0
s^1	$\frac{16-4K}{5}$	0	
s^0	K + 1		

From row of s', 16 – 4K = 0

$\therefore K_{mar} = 4$

$A(s) = 5s^2 + K + 1 = 0$

$\therefore s^2 = \frac{(4+1)}{5} = -1$

$\therefore s = \pm j$

Intersection points with $j\omega$ axis are at ± j.

Step 7 : No complex open loop poles hence angle of departure not required.

Step 8 : Root locus is shown in the Fig. Q.24.1(b).

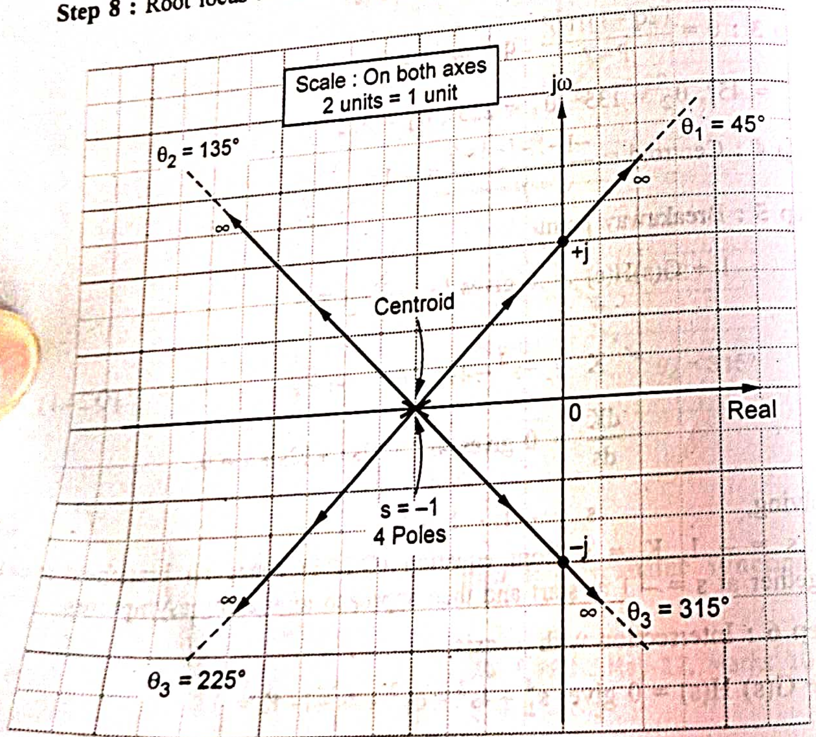


Fig. Q.24.1(b)

Q.25 Plot a root locus for the system

$$G(s)\,H(s) = \frac{K}{s(s+4)(s^2+4s+13)} \quad 0 < K < \infty.$$

[SPPU : May-22, Marks 10]

Ans. : Step 1 : P = 4, Z = 0, N = 4, P – Z = 4 branches to ∞

Starting points : s = 0, – 4, – 1 ± j3

Terminating points : s = ∞, ∞, ∞, ∞

Step 2 : Sections of real axis as shown in the Fig. Q.25.1(a).

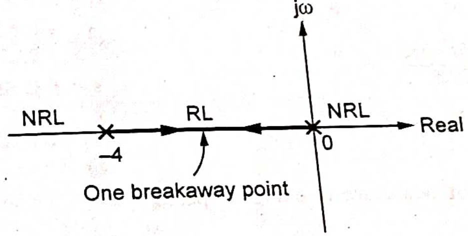


Fig. Q.25.1 (a)

Step 3 : $\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$ as obtained in Q.24 as P – Z = 4

Step 4 : Centroid $= \frac{-2-2-4-0}{4} = -2$

Step 5 : Breakaway point

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1 + \frac{K}{s(s+4)(s^2+4s+13)} = 0$$

$$\therefore s^4 + 8s^3 + 29s^2 + 52s + K = 0 \quad ...(Q.25.1)$$

$$\therefore \quad K = -s^4 - 8s^3 - 29s^2 - 52s \quad ...(Q.25.2)$$

$$\frac{dK}{ds} = 0 \text{ gives } 4s^3 + 24s^2 + 58s + 52 = 0$$

Solving, $s = -2, -2, \pm j1.58$

Using angle condition, it can be confirmed that all three are valid breakaway points. Using equation (Q.25.2) the value of K is,

K = + 36 for s = – 2 and K = + 42.25 for s = – 2 ± j1.58

Thus $s = -2$ breakaway point occurs first and then remaining two breakaway points occur simultaneously for $K = +42.25$.

Step 6 : Intersection with $j\omega$ axis

$s^4 + 8s^3 + 29s^2 + 52s + K = 0$

s^4	1	29	K
s^3	8	52	0
s^2	22.5	K	
s^1	$\frac{1170-8K}{22.5}$	0	
s^0	K		

From s^1 row, $1170 - 8K = 0$

$\therefore \quad K_{mar} = 146.25$

$A(s) = 22.5s^2 + K = 0$

$\therefore \quad s^2 = \frac{-146.25}{22.5} = -6.5$

$\therefore \quad s = \pm j2.55$

Step 7 : Angle of departure at complex poles

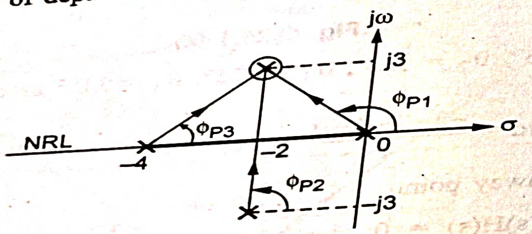


Fig. Q.25.1 (b)

$$\phi_{P1} = 180° - \tan^{-1}\frac{3}{2} = 123.69°$$

$$\phi_{P2} = 90°, \ \phi_{P3} = \tan^{-1}\frac{3}{2} = 56.31°$$

$$\therefore \quad \sum \phi_P = 123.69° + 90° + 56.31° = 270°$$

$$\therefore \quad \phi = \sum \phi_P - \sum \phi_Z = 270° - 0° = 270°$$

$$\therefore \quad \phi_d = 180° - \phi = -90° \text{ at } -2 + j3$$

and $\quad \phi_d = +90°$ at $-2 - j3$

Step 8 : The complete root locus is shown in the Fig. Q.25.1(c).

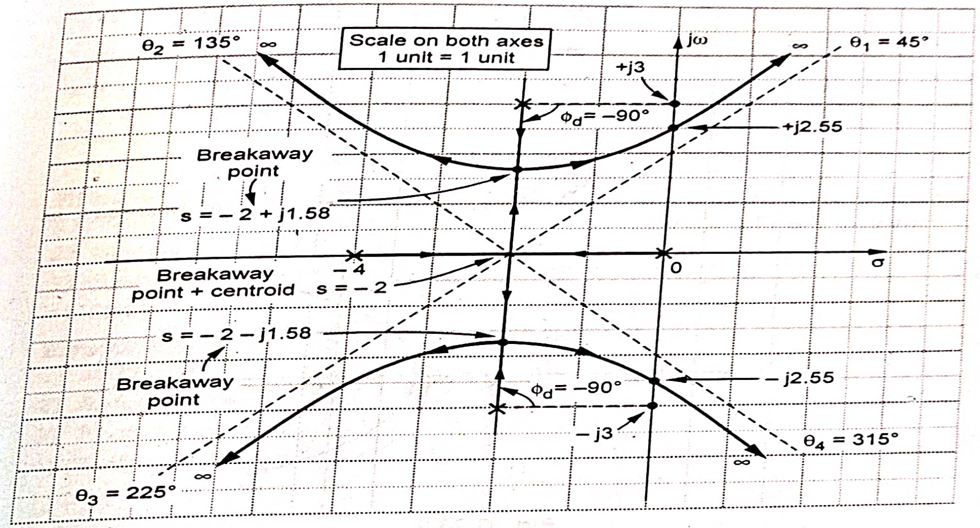


Fig. Q.25.1(c)

3.12 : Effect of Addition of Pole and Zero on Root Locus

Q.26 What are the effects adding open loop poles and zero on the nature of the root locus and on system ?

Ans.:

- Consider, $G(s)H(s) = \frac{K}{s(s+2)}$. The root locus of this G(s)H(s) is shown in the Fig. Q.26.1.
- It can be seen that for any value of 'K' system is totally stable.

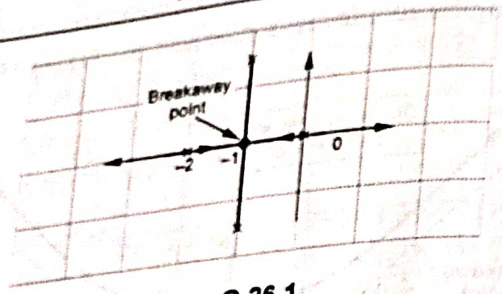


Fig. Q.26.1

- Now if pole at $s = -4$ is added to G(s)H(s) root locus becomes as shown in the Fig. Q.26.2.

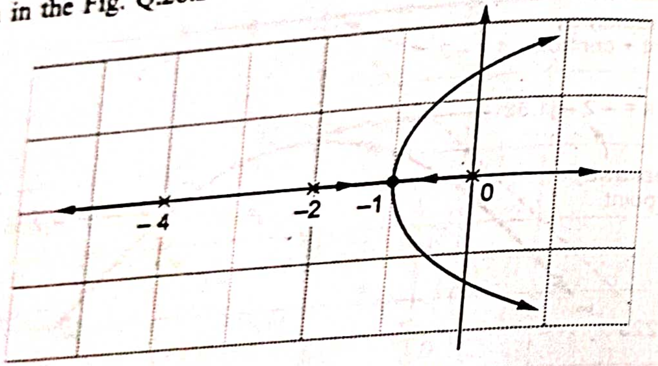


Fig. Q.26.2

- After addition of pole in left half, it can be seen that after some value of 'K', the root locus branches moves in right half of s-plane. So the system becomes conditionally stable from absoultely stable. Thus the stability of system gets restricted. The root locus shifts towards R.H.S of s-plane

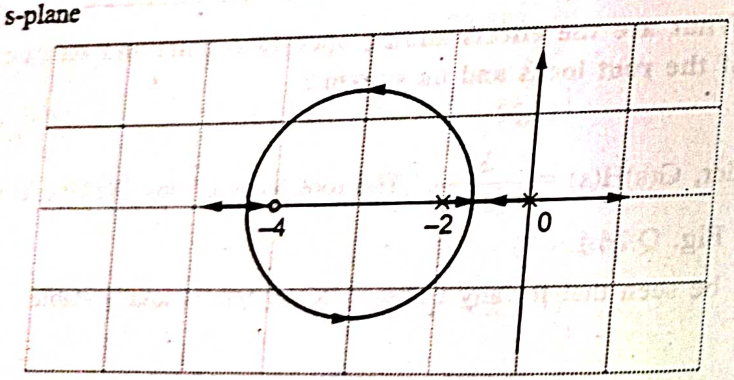


Fig. Q.26.3

- Now consider the addition of zero at $s = -4$ to the original G(s)H(s) i.e. $G(s)H(s) = \frac{K(s+4)}{s(s+2)}$ then the root locus becomes as shown in the Fig. Q.26.3.
- It can be seen that root locus shift towards left i.e. towards zero which is added. So as roots move towards left half of s-plane relative stability increases. Also it increases the range of operating values of 'K' for system stability.

Effects of addition of open loop poles can be summarized as :

1) Root locus shifts towards imaginary axis.
2) System stability relatively decreases.
3) System becomes more oscillatory in nature.
4) Range of operating values of 'K' for stability of the system decreases.

Effects of addition of open loop zeros can be summarized as :

1) Root locus shifts to left away from imaginary axis.
2) Relative stability of the system increases.
3) System becomes less oscillatory.
4) Range of operating values of 'K' for system stability increases.

END...

Unit IV

4 Frequency Domain Analysis

4.1 : Concept of Frequency Response

Q.1 Explain the concept of frequency response.

Ans. : • The steady state response of a system to a purely sinusoidal input when input frequency is varied from 0 to ∞ is defined as frequency response of a system.

• In such method, the effect of change in input frequency of the input signal is studied on the magnitude and phase of the system.

• To obtain frequency domain transfer function replace s by $j\omega$

• To get frequency response means to sketch the variation in magnitude and phase angle of $G(j\omega)$, when ω is varied from 0 to ∞.

• $G(j\omega) = M \angle \phi$ where M = Magnitude → f(ω), ϕ = Phase Angle → f(ω), ω = Input frequency.

• **Frequency response means to sketch variation in M and ϕ against ω. The stability of system can be decided from such frequency responses.**

4.2 : Derivation of Resonant Peak (M_r) and Resonant Frequency (ω_r)

Q.2 Derive the expressions for resonant peak M_r and resonant frequency ω_r for a standard second order system interms of ξ and ω_n.

[SPPU : Dec.-02, 03, 08, May-05, Marks 6]

Ans. : • Consider a standard second order system with the closed loop transfer function in time domain as,

T.F. $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

• To get frequency domain T.F. replace 's' by 'jω'.

$$\frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\xi\omega_n j\omega + \omega_n^2} = \frac{\omega_n^2}{-\omega^2 + 2\xi\omega_n j\omega + \omega_n^2}$$

• Dividing by ω_n^2, $$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right] + 2\xi j \frac{\omega}{\omega_n}}$$

• Replacing $\frac{\omega}{\omega_n} = x$, $$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{[1-x^2] + 2\xi j x}$$

• In frequency response, the second order system shows a peak. This is called resonant peak M_r and corresponding frequency is called resonant frequency ω_r.

$$\text{Magnitude } \left|\frac{C(j\omega)}{R(j\omega)}\right| = \frac{1}{\sqrt{(1-x^2)^2 + 4\xi^2 x^2}}$$

• Find the value of x which maximises the magnitude i.e. $\frac{dM}{dx} = 0$.

$$\frac{dM}{dx} = \frac{d}{dx}\left[((1-x^2)^2 + 4\xi^2 x^2)^{\frac{-1}{2}}\right]$$

$$= -\frac{1}{2}[(1-x^2)^2 + 4\xi^2 x^2]^{\frac{-3}{2}} \frac{d}{dx}[(1-x^2)^2 + 4\xi^2 x^2] = 0$$

• Solving, $4x[x^2 + 2\xi^2 - 1] = 0$ i.e. $x = 0$ or $x^2 + 2\xi^2 - 1 = 0$

$x^2 = 1 - 2\xi^2$ i.e. $x = \sqrt{1-2\xi^2}$ as x = 0 has no pratical significance

$\therefore \quad \frac{\omega}{\omega_n} = \sqrt{1-2\xi^2}$ which maximises M.

$$\therefore \quad \boxed{\omega_r = \omega_n \sqrt{1-2\xi^2}} \quad \text{... Resonant frequency}$$

• The resonant peak is obtained by subtituting ω_r in expression of M_r,

$$M_r = \frac{1}{\sqrt{\left[1-(\sqrt{1-2\xi^2})^2\right]^2 + 4\xi^2(\sqrt{1-2\xi^2})^2}}$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}}$$ ... Resonant peak

$\therefore$

Q.3 Determine damping factor, undamped natural frequency, resonant peak and resonant frequency for the system with closed loop transfer function :

$$G(s) = \frac{100}{s^2+10s+100}.$$

☞ [SPPU : May-17, Marks 4]

Ans. : Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$

$\omega_n^2 = 100$ i.e. $\omega_n = 10$ rad/sec

$2\xi\omega_n = 10$ i.e $\xi = 0.5$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \frac{1}{2\times 0.5\sqrt{1-0.5^2}} = 1.1547$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 10\sqrt{1-2\times 0.52} = 7.071 \text{ rad/sec}$$

Q.4 Determine damping factor, undamped natural frequency, resonant peak and resonant frequency for the system with closed loop transfer function :

$$G(s) = \frac{36}{s^2+6s+36}$$

☞ [SPPU : Dec.-17, Marks 4]

Ans. : Compare denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$\omega_n^2 = 36$, $\omega_n = 6$ rad/sec

$2\xi\omega_n = 6$ i.e. $\xi = \frac{6}{2\times 6} = 0.5$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = 1.154$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 4.243 \text{ rad/sec}$$

Q.5 For the system with closed loop transfer function $G_{CL}(s) = \frac{400}{s^2+20s+400}$, determine resonant peak, resonant frequency, damping factor and natural frequency.

☞ [SPPU : May-18, Marks 4]

Ans. : Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$\omega_n^2 = 400$ i.e. $\omega_n = 20$ rad/sec

$2\xi\omega_n = 20$ i.e. $\xi = \frac{20}{2\times 20} = 0.5$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = 1.1547$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = 14.142 \text{ rad/sec}$$

Q.6 Determine resonant peak (M_r) and resonant frequency (ω_r) for the unity feedback system with open loop transfer function :

$$G(s) = \frac{9}{s(s+4)}$$

☞ [SPPU : Dec.-18, Marks 4]

Ans. : H(s) = 1

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{9}{s(s+4)}}{1+\frac{9}{s(s+4)}} = \frac{9}{s^2+4s+9}$$

Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$,

$\omega_n^2 = 9$, $\omega_n = 3$ rad/sec, $2\xi\omega_n = 4$, $\xi = 0.667$

$$\therefore M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = 1.006, \quad \omega_r = \omega_n\sqrt{1-2\xi^2} = 0.996 \text{ rad/sec.}$$

Q.7 For unity feedback system with open loop transfer function $G(s) = \frac{100}{s(s+9)}$. Determine damping factor, undamped natural frequency, resonant peak, resonant frequency.

☞ [SPPU : May-19, Marks 4]

Ans. : The characteristic equation is $1 + G(s)H(s) = 0$

$$1+\frac{100}{s(s+9)} = 0 \text{ i.e. } s^2+9s+100 = 0 \text{ i.e. } s^2+2\xi\omega_n s+\omega_n^2 = 0$$

$$\therefore \quad \omega_n^2 = 100,\ \omega_n = 10 \text{ rad/sec and } 2\xi\omega_n = 9,\ \xi = 0.45$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 10\sqrt{1-(0.45)^2} = \mathbf{8.93\ rad/sec}$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \mathbf{1.244}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = \mathbf{7.714\ rad/sec}$$

Q.8 For the system with closed loop transfer function :

$$G(s) = \frac{25}{s^2+6s+25}.$$

Determine resonant peak, resonant frequency, damping factor and natural frequency. **[SPPU : Dec.-19, Makrs 4]**

Ans. : Comparing denominator with $s^2+2\xi\omega_n s+\omega_n^2$

$$\omega_n^2 = 25,\ \omega_n = 5 \text{ and } 2\xi\omega_n = 6$$

$$\therefore \quad \xi = \frac{6}{2\omega_n} = \frac{6}{2\times 5} = \mathbf{0.6},\quad \omega_n = \mathbf{5\ rad/sec}$$

$$\therefore \quad M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} = \mathbf{1.042}$$

$$\omega_r = \omega_n\sqrt{1-2\xi^2} = \mathbf{2.645\ rad/sec}$$

4.3 : Co-relation between Time and Frequency Domain

Q.9 Explain the co-relation between time and frequency domain.
[SPPU : Dec.-98, 02, 03, May-02, 04, 07, 08, 11, Marks 7]

Ans. : The two important parameters associated with a standard second order system in frequency domain are,

i) Resonant frequency (ω_r) : This is the fequency at which the system shows a peak in its frequency response. It is given by,

$$\omega_r = \omega_n\sqrt{1-2\xi^2}.$$

ii) Resonant peak (M_r) : It is the peak value of the frequency response of a second order system ocurring at resonant frequency ω_r and given by, $M_r = \dfrac{1}{2\xi\sqrt{1-\xi^2}}$.

- While in time domain it is known that for a standard second order system, damped frequency is $\omega_d = \omega_n\sqrt{1-\xi^2}$ and $M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$.
- From these expressions co-relation between frequency domain and time domain can be obtained as,

1) Both M_p and M_r are the functions of the damping ratio alone. So both indicates the relative stability of the system.

2) As 'ξ' increases, M_p decreases and gets vanished when $\xi = 1$. After that, system does not produce any overshoot.

While in frequency domain M_r will vanish if, $\sqrt{1-2\xi^2} = 0$

i.e. $2\xi^2 = 1$ i.e. $\xi^2 = 1/2$ $\therefore \xi = 0.707$

And in such case system will not exhibit resonant peak. This can be shown in Fig. Q.9.1.

3) When ξ is very small i.e. less than 0.4, both M_p and M_r will be very large and are not desirable. So 'ξ' is generally designed to be between $0.4 < \xi < 0.707$ where M_p and M_r are comparable to each other.

4) As $\xi \to 0$, M_p achieves 1 i.e. 100 % overshoot while M_r tries to approach to ∞ and hence both are undesirable from system point of view.

5) Also when ξ is between 0.4 to 0.707 both ω_r and ω_d are comparable to each other while when $\xi \to 0$, both ω_r and ω_d approaches the value ω_n. When 'ξ' is small, ω_d is large and hence rise time is small.

Similarly the ξ is small ω_r is large and hence system is more fast in the response. Hence ω_r indicates speed of the response.

- All these co-relations are shown in the Fig. Q.9.1, Q.9.2, Q.9.3 and Q.9.4.

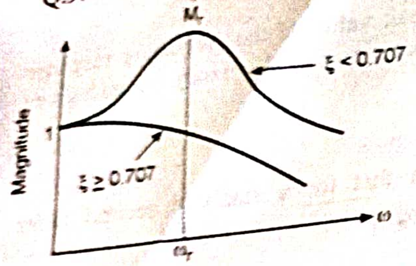


Fig. Q.9.1

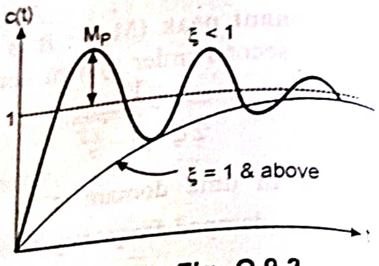


Fig. Q.9.2

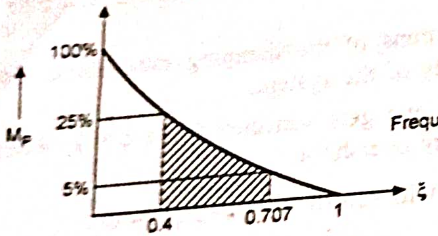


Fig. Q.9.3

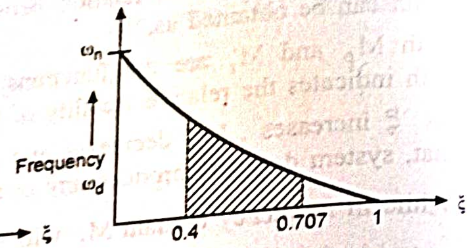


Fig. Q.9.4

- The two responses are comparable between $0.4 < \xi < 0.707$.

Bode Plot

4.4 : Introduction to Bode Plot

Q.10 What is Bode plot ?

Ans. : • Bode plot consists of two plots which are,

1) Magnitude expressed in logarithmic values against logarithmic values of frequency called Magnitude plot.

2) Phase angle in degrees against logarithmic values of frequency called Phase angle plot.

The magnitude M $= |G(j\omega)H(j\omega)| = 20 \text{ Log}_{10} |G(j\omega)|$ dB

The phase angle $\phi = \angle G(j\omega)H(j\omega)$ in degrees

- The two plots are shown in the Fig. Q.10.1.

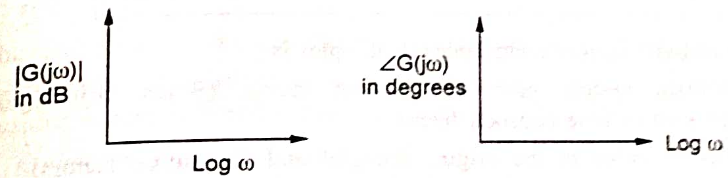


Fig. Q.10.1 Magnitude plot and Phase plot

- Both these plots are sktched on the same semi-log paper and together called Bode plot of the system.

Important Points to Remember

- A semi-log paper is used to sketch the Bode plot. In such paper the X-axis is divided into a logarithmic scale which is non linear one. While Y-axis is divided into linear scale and hence it is called ***semilog paper.***
- The logarithmic scale available on X-axis takes care of logarithmic value of ω There is no need to find the logarithmic values of frequency ω

4.5 : Standard Form and Factors of G(jω)H(jω)

Q.11 What is the standard form of G(jω)H(jω) ? State the general factors of G(jω)H(jω) which contribute to the Bode plot.

Ans. : The standarad form of G(jω)H(jω) is called time constant form and given by,

$$G(s)H(s) = \frac{K(1+T_1 s)(1+T_2 s) \ldots\ldots}{s^p (1+T_a s)(1+T_b s)}$$

where K = Resultant system gain p = Type of the system

$T_1, T_2, T_a, T_b ......$ = Time constants of different poles and zeros.

- Frequency domain transfer function can be obtained by substituting $s = j\omega$ in the above expression,

$$G(j\omega)H(j\omega) = \frac{K(1+T_1 j\omega)\,(1+T_2 j\omega)\,.....}{(j\omega)^p\,(1+T_a j\omega)\,(1+T_b j\omega)\,.....}$$

- List of basic factors contributing Bode plot is ,

1) Resultant system gain K, constant factor. (When $G(j\omega)H(j\omega)$ is expressed in time constant form).

2) Poles or zeros at the origin. (Integral and Derivative factors) i.e. $(j\omega)^{\pm p}$. Either poles or zeros at origin will be present.

3) Simple poles and zeros also called first order factors of the form $(1+sT)^{\pm 1}$ i.e. $(1+j\omega T)^{\pm 1}$

4) Quadratic factors i.e. quadratic pole or zero, which cannot be factorised into real factors i.e. having complex conjugate roots, of the form $\left(1+\frac{2\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}\right)^{\pm 1} = \left[1+2\xi j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]^{\pm 1}$

Important Points to Remember

1. **Constant K :** The magnitude plot of K is a straight line parallel to X-axis i.e. Logω axis at a distance of (20 Log K) dB above or below 0 dB line depending on the value of K. Postive K does not contribute to phase angle plot.

2. **Poles at the origin :** One pole at the origin contributes a line of slope – 20 dB/decade passing through intersection point of $\omega = 1$ and 0 dB lines. If there are 2 poles at the origin, the slope of the straight line changes to – 40 dB/decade and so on. In phase angle plot, each pole at the origin contributes fixed angle of $-90°$ at all the frequencies. Two poles at the origin contibute $-180°$ and so on. For zeros at origin the sign of the slope changes to positive in magnitude plot while the sign of the angle changes to positive in phase angle plot.

3. **Simple Poles or Zeros $(1+Ts)^{\pm 1}$** i.e. $(1+j\omega T)^{\pm 1}$: Simple pole contributes a straight line of slope – 20 dB/dec after its corner frequency which is given by $\omega_C = 1/T$. Till $\omega = \omega_C = 1/T$, the factor does not contribute to magnitude plot. For a simple zero, the sign of the slope changes to positive, the nature remains same.

- The phase angle of such factors is given by $\pm\tan^{-1}\omega T$ and its contribution is required to be calculated at various frequencies. The + sign for simple zero and – sign for simple pole.

4. **Quadratic Poles or Zeros :**

- These are represented as

$$\left(1+\frac{2\xi}{\omega_n}s+\frac{s^2}{\omega_n^2}\right)^{\pm 1} \text{ i.e } \left[1+2\xi j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]^{\pm 1}$$

- The corner frequency of such factors is given by, $\omega_C = \omega_n$.
- Quadratic pole contributes a straight line of slope – 40 dB/dec after its corner frequency which is given by $\omega_C = \omega_n$. Till $\omega = \omega_C$, it does not contribute to magnitude plot. For a quadratic zero, the sign of the slope changes to positive, the nature remains same.
- The phase angle of such factor is given by,

$\phi = -\tan^{-1}\left\{\frac{2\xi(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right\}$ The positive sign is to be used if the factor is quadratic zero.

- **When the magnitude plot of two factors representing straight lines are to be added together in the Bode plot, then resultant line always has a slope which is algebraic addition of the individual slopes of the two lines which are to be added.**
- **The starting slope of the Bode plot for the function G(s)H(s) gets decided by number of poles or zeros at origin present in G(s)H(s).**

4.6 : Frequency Response Specifications

Q.12 Define the following frequency domain specifications : i) Resonant peak ii) Resonant frequency iii) G.M. iv) P.M. v) Gain cross-over frequency vi) Phase cross-over frequency

[SPPU : May-01,04,12,22, Dec.-01,03,05,06,10,11,13, Marks 6]

Ans. : i) Resonant peak (M_r) : It is the maximum value of magnitude of the closed loop frequency response. Larger the value of resonant peak more is the value of peak overshoot of system for step input.

ii) Resonant frequency (ω_r) : The frequency at which resonant peak M_r occurs in closed loop frequency response is called resonant frequency.

iii) Gain crossover frequency (ω_{gc}) : The frequency at which magnitude of $G(j\omega)H(j\omega)$ is unity i.e. 1 is called **gain crossover frequency.**

- Generally magnitude of $G(j\omega)H(j\omega)$ is expressed in dB. And dB value of 1 is $20 Log_{10} 1 = 0$ dB.
- Thus gain cross-over frequency is the frequency at which magnitude plot crosses the 0 dB line. Hence at ω_{gc}, magnitude of $G(j\omega)H(j\omega)$ is 0 dB.

iv) Phase crossover frequency (ω_{pc}) : The frequency at which phase angle of $G(j\omega)H(j\omega)$ is $-180°$ is called **phase crossover frequency,** ω_{pc}.

v) Gain margin G.M. : As gain 'K' is increased, the system stability reduces and for a certain value of 'K' it becomes marginally stable.

- Gain margin is defined as the margin in gain K allowable by which gain can be increased till system reaches on the verge of instability.
- Mathematically it can be defined as reciprocal of the magnitude of the $G(j\omega)H(j\omega)$ measured at phase crossover frequency.

$$\therefore \quad G.M. = \frac{1}{|G(j\omega)H(j\omega)|_{\omega = \omega_{pc}}}$$

In decibels

$$G.M. = 20 \text{ Log} \frac{1}{|G(j\omega)H(j\omega)|_{\omega = \omega_{pc}}}$$

$$G.M. = -20 \text{ Log}_{10} |G(j\omega)H(j\omega)|_{\omega = \omega_{pc}}$$

- More positive the G.M., more stable is the system.

vi) Phase margin P.M. : Similar to the gain, it is possible to introduce phase lag in the system i.e. negative angles without affecting magnitude plot of $G(j\omega)H(j\omega)$.

- The amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability is called **phase margin P.M.**
- Mathematically it can be defined as,

$$P.M. = [\angle G(j\omega)H(j\omega)|_{at\ \omega = \omega_{gc}}] - [(-180°)]$$

$$\therefore \quad P.M. = 180° + \angle G(j\omega)H(j\omega)|_{\omega = \omega_{gc}}$$

- Positive P.M. indicates stable system while negative P.M. indicates unstable system. More positive the P.M., more stable is the system.

4.7 : G.M. and P.M. from Bode Plot

Q.13 Define G.M. and P.M. How to find them from Bode plot ?

[SPPU : Dec.-10, 12, May-13, Marks 6]

OR How to determine stability from Bode plot ?

[SPPU : Dec.-13, Marks 6]

Ans. : For definitions of G.M. and P.M., refer answer of Q.12.

- In a Bode plot, identify $\omega = \omega_{pc}$ and extend $\omega = \omega_{pc}$ line upwards till it intersects resultant magnitude plot at point A. The magnitude corresponding to point A is $|G(j\omega)H(j\omega)|_{\omega = \omega_{pc}}$.
- The difference between 0 dB and magnitude corresponding to point A i.e. $|G(j\omega)H(j\omega)|_{\omega = \omega_{pc}}$ is **Gain Margin.** If point A is below 0 dB, G.M. is positive and if point A is above 0 dB, G.M. is negative.
- For phase margin, identify $\omega = \omega_{gc}$ and extend $\omega = \omega_{gc}$ line downwards till it intersects phase angle plot at point C. The angle corresponding to the point C is $\angle |G(j\omega)H(j\omega)|_{\omega = \omega_{gc}}$.

- The distance between this point C i.e. $\angle|G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$ and $-180°$ line is nothing but the phase margin. If point C is above $-180°$ line, P.M. is positive and if it is below $-180°$ line, P.M. is negative.
- System is said to be stable when P.M. and G.M. are positive while system is said to be unstable when both P.M. and G.M. are negative. Now when system is on the verge of instability i.e. marginally stable in nature then G.M. and P.M. both are zero. This is possible when $\omega_{gc} = \omega_{pc}$.
- The G.M. and P.M. for stable system is shown in the Fig. Q.13.1.

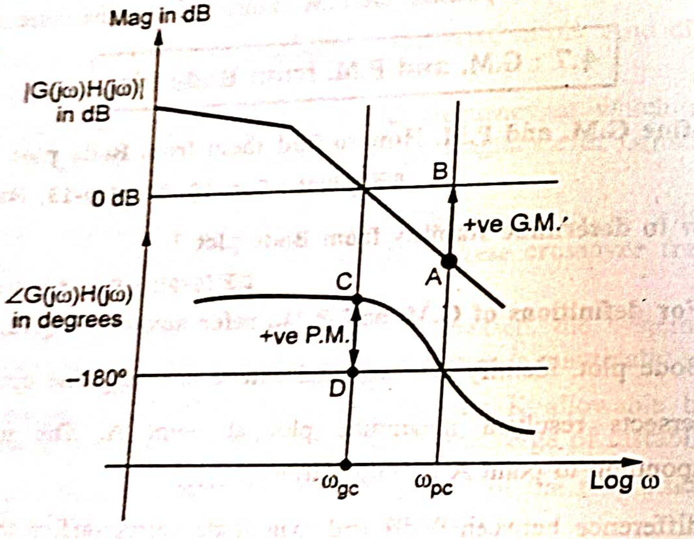


Fig. Q.13.1 $\omega_{gc} < \omega_{pc}$, G.M. and P.M. positive, stable system

4.8 : Steps to Sketch Bode Plot

Important Points to Remember

1) Express given G(s)H(s) into time constant form.
2) Draw a line of 20 Log K dB which is parallel to Log ω axis i.e. X-axis.

3) Draw a line of appropriate slope representing poles or zeros at the origin, passing through intersection point of $\omega = 1$ and 0 dB. For 1 pole at the origin – 20 dB/dec, for 2 poles at the origin – 40 dB/dec and so on.
4) Shift the intersection point of $\omega = 1$ and 0 dB on 20 Log K line and draw parallel line to the line drawn in step 3. This is addition of constant K and number of poles or zeros at the origin.
5) Change the slope of this line at various corner frequencies by appropriate value i.e. depending upon which factor is occurring at corner frequency. For a simple pole, slope must be changed by – 20 dB/decade, for a simple zero by + 20 dB/decade etc. **Do not draw these individual lines.** Change the slope of line obtained in step 5 by respective value and draw line with resultant slope. Continue this line till it intersects next corner frequency line. Change the slope and continue. Apply necessary correction for quadratic factor if required.
6) Prepare the phase angle table and obtain the table of ω and resultant phase angle ϕ_R by actual calculation. Plot these points and draw the smooth curve obtaining the necessary phase angle plot.

- **Remember that at every corner frequency slope of resultant line must change.**

7) Determine ω_{gc}, ω_{pc}, G.M. and P.M. from the Bode plot and predict the stability of the system.

Q.14 For the unity feedback system with open loop transfer function. $G(s) = \frac{50}{s(s+2)(s+10)}$, ***sketch Bode plot.***

Determine gain crossover frequency, phase crossover frequency, gain margin and phase margin. Also investigate the stability.

[SPPU : Dec.-17, Makrs 8]

Ans. : Step 1 : $G(s)H(s) = \frac{2.5}{s(1+0.5s)(1+0.1s)}$ ... Time constant form

Step 2 : Factors

i) K = 2.5, 20 Log K = 8 dB, straight line parallel to Log ω axis.

ii) $\frac{1}{s}$, one pole at origin, straight line of slope – 20 dB/dec.

Passing through intersection of ω = 1 and 0 dB

iii) $\frac{1}{1+0.5\,s}$, simple pole, $T_1 = 0.5$, $\omega_{C1} = \frac{1}{T_1} = 2$, straight line of slope – 20 dB/dec for ω ≥ 2

iv) $\frac{1}{1+0.1\,s}$, simple pole, $T_2 = 0.1$, $\omega_{C2} = \frac{1}{T_2} = 10$, straight line of slope – 20 dB/dec for ω ≥ 10

Range of ω	0 < ω < 2	2 ≤ ω < 10	10 ≤ ω < ∞
Resultant slope in dB/dec	– 20	– 20 – 20 = – 40	– 40 – 20 = – 60

Step 3 : Phase angle table

$$G(j\omega)H(j\omega) = \frac{2.5}{j\omega(1+0.5j\omega)(1+0.1j\omega)}$$

ω	$\frac{1}{j\omega}$	$-\tan^{-1} 0.5\omega$	$-\tan^{-1} 0.1\omega$	ϕ_R
0.2	– 90°	– 5.71°	– 1.14°	– 96.85°
2	– 90°	– 45°	– 11.3°	– 146.3°
5	– 90°	– 68.19°	– 26.56°	– 184.7°
10	– 90°	– 78.69°	– 45°	– 213.7°
∞	– 90°	– 90°	– 90°	– 270°

Step 4 : The Bode plot and the answers are given in the Fig. Q.14.1.

(See Fig. Q.14.1 on next page)

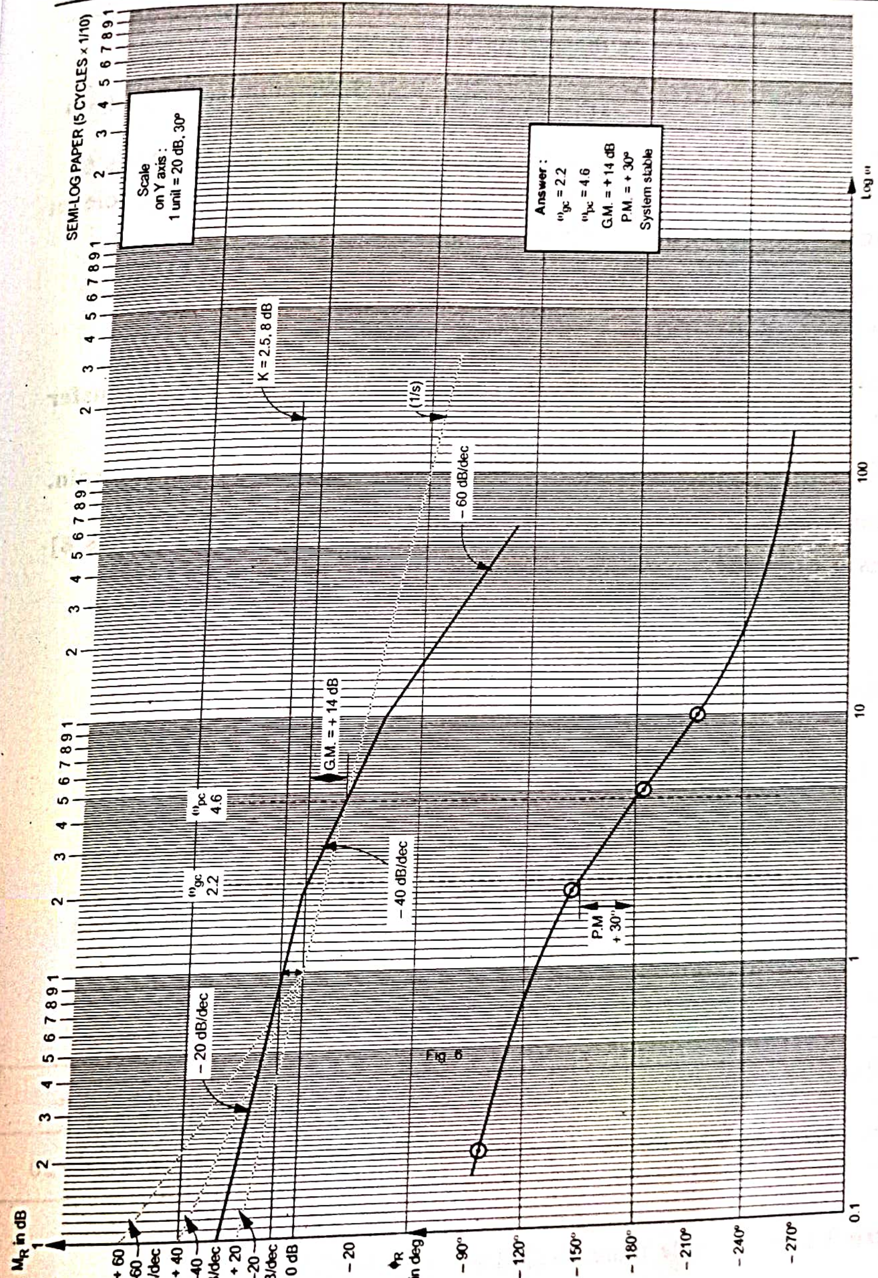


Fig. Q.14.1

Q.15 Draw Bode plot of the system with open loop transfer function $G(s) = \dfrac{50}{s(s+5)(s+10)}$ and determine ω_{gc}, ω_{pc}, gain margin and phase margin. **[SPPU : May-18, Dec.-19, Marks 8]**

Ans. : Refer Q.14 for the procedure. Factors are, K = 1, one pole at origin $\frac{1}{s}$ and simple poles with $\omega_{C1} = 5$, $\omega_{C2} = 10$.

Verify the answers as : ω_{gc} = **1 rad/sec,** ω_{pc} = **7 rad/sec**

G.M. = + **24 dB**, P.M. = + **73°**. System is stable.

Q.16 Draw Bode plot of the system with open loop transfer function :

$$G(s) = \frac{20(s+5)}{s(s+10)}$$ **and determine gain margin, phase margin. Also comment on stability.** **[SPPU : Dec.-18, May-22, Marks 8]**

Ans. : Step 1 : Time constant form,

$$G(s) = \frac{10(1+0.2s)}{s(1+0.1s)}$$

Step 2 : Factors

i) K = 10, 20 log K = 20 dB, straight line parallel to log ω axis.

ii) One pole at the origin, straight line of slope – 20 dB/dec

iii) Simple zero, 1 + 0.2s, $\omega_{C1} = \frac{1}{0.2} = 5$,

straight line of slope + 20 dB/dec for ω≥5

iv) Simple pole, $\frac{1}{1+0.1\,s}$, $\omega_{C2} = \frac{1}{0.1} = 10$,

straight line of slope – 20 dB/dec for ω≥10

Range of ω	0 < ω < 5	5≤ ω < 10	10≤ ω < ∞
Resultant slope in dB/dec	– 20	–20 + 20 = 0	0 – 20 = –20

Step 3 : Phase angle table, $G(j\omega) = \dfrac{10(1+0.2\,j\omega)}{j\omega(1+0.1\,j\omega)}$

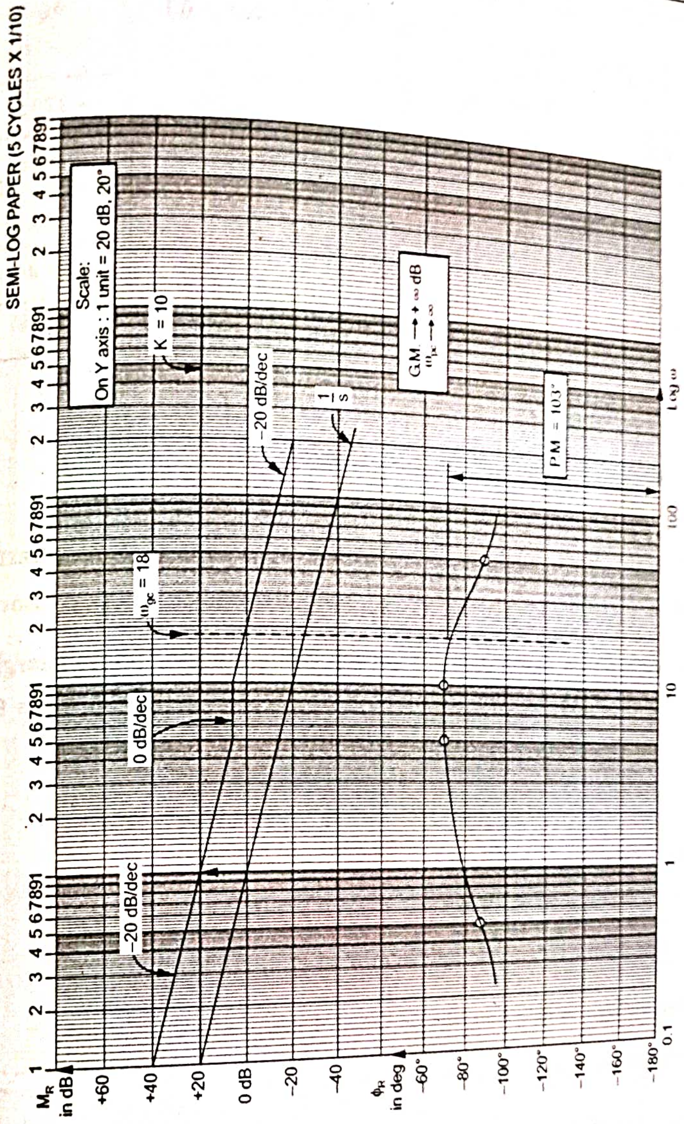


Fig. Q.16.1

ω	$\frac{1}{j\omega}$	$+\tan^{-1} 0.2\omega$	$-\tan^{-1} 0.1\omega$	ϕ_R
0.5	$-90°$	$+5.71°$	$-2.86°$	$-87.15°$
5	$-90°$	$+45°$	$-26.56°$	$-71.56°$
10	$-90°$	$+63.43°$	$-45°$	$-71.57°$
50	$-90°$	$+84.23°$	$-78.69°$	$-84.46°$
∞	$-90°$	$+90°$	$-90°$	$-90°$

Step 4 : The Bode plot is shown in the Fig. Q.16.1.

From the plot, G.M. = + ∞ dB, P.M. = **103°**

Hence the system is **stable** in nature.

Q.17 Draw Bode plot of the system with open loop transfer function : $G(s) = \frac{40}{s(s+2)(s+20)}$ and determine gain crossover frequency, phase cross over frequency, gain margin, phase margin. Also comment on stability. **[SPPU : May-19, Marks 8]**

Ans. : Step 1 : G(s)H(s) in time constant form.

$$\therefore \quad G(s)H(s) = \frac{1}{s(1+0.5s)(1+0.05s)}$$

Step 2 : Factors

i) K = 1, 20 log K = 0 dB

ii) $\frac{1}{s}$, one pole at origin, straight line of slope – 20 dB/dsec passing through intersection of ω = 1 and 0 dB

iii) Simple pole, $\frac{1}{1+0.5s}$, $T_1 = 0.5$, $\omega_{C1} = \frac{1}{T_1} = 2$, straight line of slope – 20 dB/dsec

iv) Simple pole, $\frac{1}{1+0.05s}$, $T_2 = 0.05$, $\omega_{C2} = \frac{1}{T_2} = 20$,

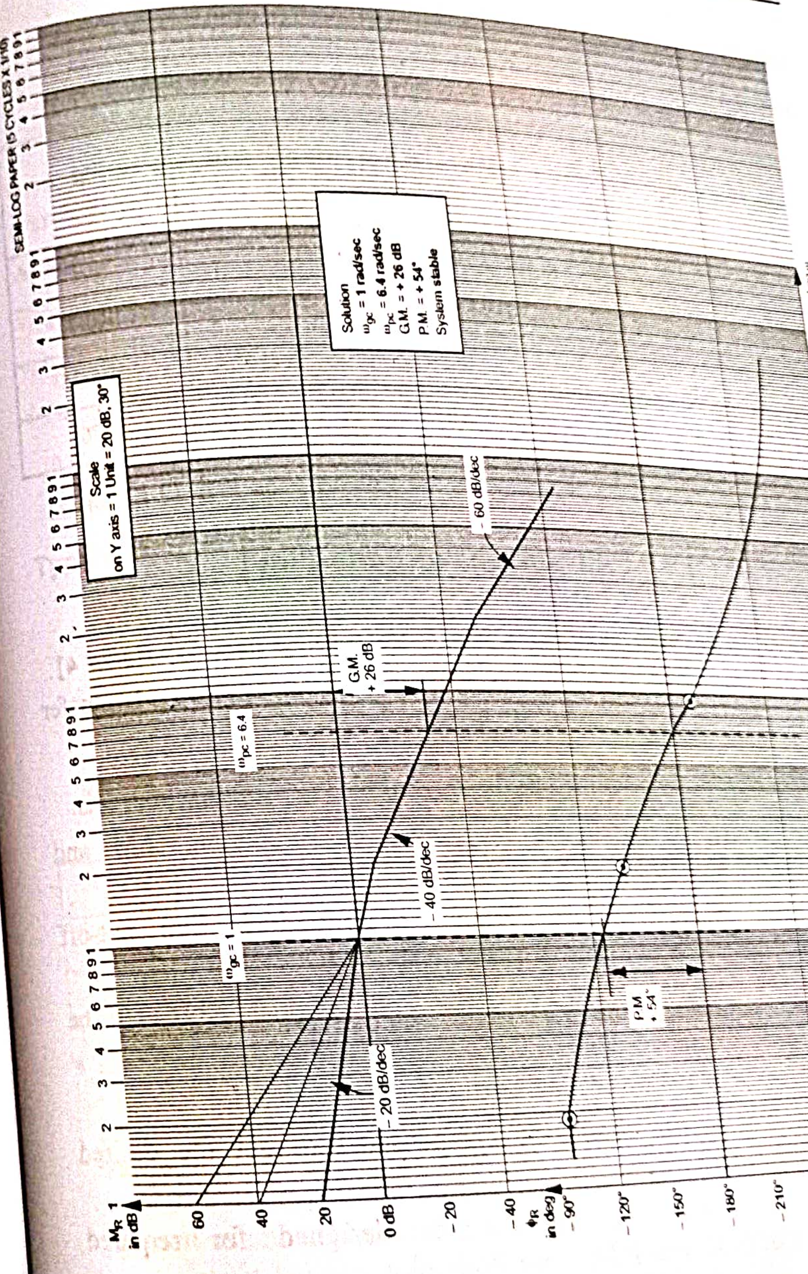


Fig. Q.17.1

straight line of slope – 20 dB/dsec

Step 3 : Phase angle table : $G(j\omega) = \frac{1}{j\omega(1+0.5j\omega)(1+0.05j\omega)}$

ω	$\frac{1}{j\omega}$	$-\tan^{-1}0.5\omega$	$-\tan^{-1}0.05\omega$	ϕ_R
0.2	– 90°	– 5.71°	– 0.57°	– 96.28°
2	– 90°	– 45°	– 5.71°	– 140.71°
10	– 90°	– 78.69°	– 26.56°	– 195.25°
∞	– 90°	– 90°	– 90°	– 270°

Step 4 : Bode plot and all the answers are given in the Fig. Q.17.1.

4.9 : Advantages of Bode Plot

Q.18 Give the advantages of Bode plot. [SPPU : Dec.-11, Marks 4]

Ans. : 1) It shows both low and high frequency characteristics of transfer function in single diagram.

2) The plots can be easily constructed using some valid approximations.

3) Relative stability of system can be studied by calculating G.M. and P.M. from the Bode plot.

4) The various other frequency domain specifications like cut-off frequency, bandwidth etc. can be determined.

5) Data for constructing complicated polar and Nyquist plots can be easily obtained from Bode plot.

6) Transfer function of system can be obtained from Bode plot.

7) It indicates how system should be compensated to get the desired response.

8) The value of system gain K can be designed for required specifications of G.M. and P.M. from Bode plot.

9) Without the knowledge of the transfer function the Bode plot of stable open loop system can be obtained experimentally.

Polar and Nyquist Plot

4.10 : Polar Plot

Q.19 What is polar plot ? Explain polar plots for Type 0, 1 and 2 systems. [SPPU : Dec.-17, Marks 4]

Ans. : • Polar plot is defined as the locus of tips of the phasors of various magnitudes plotted at the corresponding phase angles for different values of frequencies from 0 to ∞.

• The polar plot starts at point representing magnitude and phase angle for $\omega = 0$. While it terminates at a point representing magnitude and phase angle for $\omega = \infty$.

Type 0 system : Consider a open loop transfer function

$$G(s)H(s) = \frac{1}{1+Ts}$$

$$G(j\omega)H(j\omega) = \frac{1}{1+Tj\omega} = \frac{1+j0}{1+j\omega T}$$

$$|G(j\omega)H(j\omega)| = M = \frac{1}{\sqrt{1+\omega^2 T^2}}, \quad \angle G(j\omega)H(j\omega) = \phi = -\tan^{-1}(\omega T)$$

ω	M	ϕ
0	1	0 °
∞	0	– 90 °

• The rotation of the plot is $-90°-0° = -90°$ i.e. 90° clockwise. So the polar plot is starting from $1\angle 0°$ it ends at $0\angle -90°$ with 90° clockwise rotation as shown in the Fig. Q.19.1.

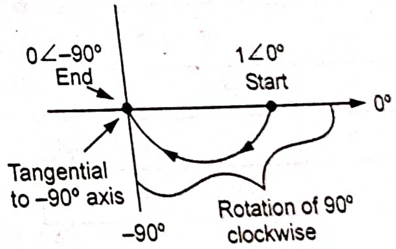


Fig. Q.19.1

Type 1 system : Consider a open loop transfer function

$$G(s)H(s) = \frac{1}{s(1+Ts)}$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+Tj\omega)} = \frac{1+j0}{(0+j\omega)(1+Tj\omega)}$$

$$\therefore \quad M = \frac{1}{\omega \times \sqrt{1+\omega^2 T^2}}, \quad \phi = -90° - \tan^{-1} \omega T$$

ω	M	φ
0	∞	– 90 °
∞	0	– 180°

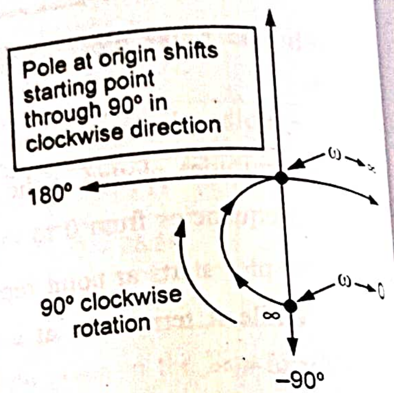


Fig. Q.19.2

- The rotation of the plot is, $-180° - (-90°) = -90°$ i.e. 90° clockwise.
- So the polar plot is starting from $\infty \angle -90°$ it ends at $0 \angle -180°$ with 90° clockwise rotation as shown in the Fig. Q.19.2.

Type 2 system : Consider a open loop transfer function

$$G(s)H(s) = \frac{1}{s^2(1+Ts)}$$

$$G(j\omega)H(j\omega) = \frac{1}{j\omega \cdot j\omega \cdot (1+Tj\omega)} = \frac{(1+j0)}{(0+j\omega)(0+j\omega)(1+j\omega T)}$$

$$\therefore \quad M = \frac{1}{\omega^2 \cdot \sqrt{1+T^2\omega^2}}, \quad \phi = -180° - \tan^{-1} \omega T$$

ω	M	φ
0	∞	– 180°
∞	0	– 270°

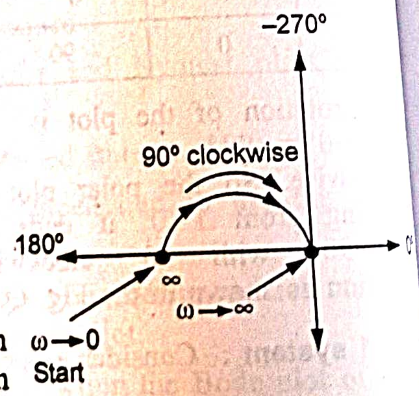


Fig. Q.19.3

- The rotation of the plot is, $-270° - (-180°) = -90°$ i.e. 90° clockwise.
- So the polar plot is starting from $\infty \angle -180°$ it ends at $0 \angle -270°$ with 90° clockwise rotation as shown in the Fig. Q.19.3.

4.11 : Nyquist Plot

Q.20 State and explain Nyquist stability criterion.

[SPPU : Dec.-2000,02,06,22, May-04,08,12, Marks 6]

Ans. : Nyquist suggested to select a single valued function F(s) as 1+ G(s)H(s) where G(s)H(s) is open loop transfer function of the system.

$$F(s) = 1 + G(s)H(s)$$

- Poles of G(s)H(s) are the open loop poles which are known as G(s)H(s) is known but zeros of 1 + G(s)H(s) are closed loop poles of the system which are not known. The stability depends on the locations of these zeros of 1 + G(s)H(s) in s-plane.
- To examine whether any of these zeros are located in right half of s-plane or not, Nyquist has suggested to select a τ(s) path which will encircle the entire right half of s-plane.
- Such a path should start from $s = +j\infty$. It should be continued till $s = -j\infty$ along imaginary axis and should be completed with a semicircle of radius ∞, encircling entire right half of s-plane as shown in the Fig. Q.20.1. This path is called **Nyquist path** and should not be changed except small modifications, while analyzing stability of any system.

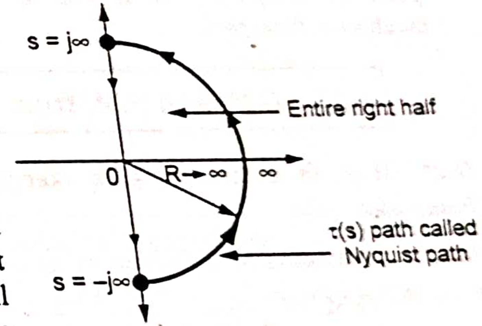


Fig. Q.20.1

- As poles of G(s)H(s) are known which are the poles of 1 + G(s)H(s), we know the value of P i.e. poles of 1 + G(s)H(s) which are encircled by Nyquist path.
- Map all the points on the Nyquist path into F -plane with the help of mapping function 1 + G(s)H(s) to get τ′(s) locus.
- **This mapped locus obtained in F-plane by mapping all the points on Nyquist path is called Nyquist plot.**
- As this locus is obtained, we can determine the number of encirclements of origin by Nyquist plot in F-plane, say N.

- As per Mapping theorem, N and P must satisfy the equation, N = Z − P and as N and P are known, we can get Z,
- Z = Number of zeros of 1 + G(s)H(s) encircled by Nyquist path in s-plane which are the closed loop poles of the system.
- But as Nyquist path encircles only right half of s-plane, Z = Number of zeros of 1 + G(s)H(s) which are located in right half of s-plane.
- For absolute stability, no zero of 1 + G(s) H(s) must be in right half of s-plane i.e. Z = 0 for stability.
- So Nyquist stability criterion is obtained by substituting Z = 0 in the equation N = Z − P.

$$N = -P \quad \text{...Nyquist stability criterion}$$

- **Nyquist stability criterion states that for absolute stability of the system, the number of encirclements of new origin of F-plane by Nyquist plot must be equal to number of poles of 1 + G(s)H(s) i.e. poles of G(s)H(s) which are in the right half of s-plane and in clockwise direction.**

4.12 : G.M. and P.M. from Polar and Nyquist Plot

Q.21 How to determine gain margin and phase margin from the Polar plot.

Ans. : According to definition of gain margin,

$$\text{G.M.} = \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

- In polar plot $|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}$ is nothing but l(OQ) where Q is the intersection of polar plot with negative real axis. Q is the point corresponding to $\omega = \omega_{pc}$.

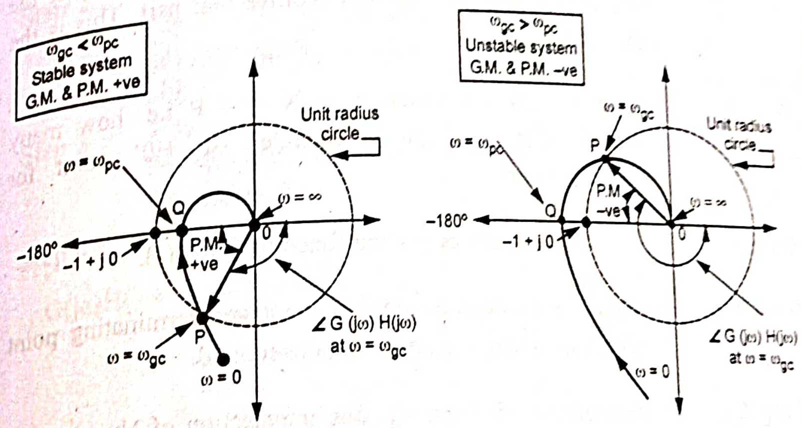
$$\therefore \quad \text{G.M.} = \frac{1}{l(\text{OQ})}$$

where Q is intersection of polar plot with negative real axis.

- According to definition of phase margin,

$$\text{P.M.} = 180° + \angle G(j\omega)H(j\omega)|_{\omega=\omega_{gc}}$$

- Draw the unit radius circle on the polar plot. The point where it intersects polar plot is denoted as P at which $|G(j\omega)H(j\omega)| = 1$ and the corresponding frequency i nothing but $\omega = \omega_{gc}$.
- So if ϕ is the angle of point P corresponding to $\omega = \omega_{gc}$ the P.M. is the angle subtended by the phasor OP with negative real axis. This is shown in the Fig. Q.21.1.



(a) Fig. Q.21.1 (b)

Important Points to Remember

To find ω_{pc} and intersection of polar plot with negative real axis

i) Rationalize the open loop transfer function G(jω)H(jω).

ii) Separate real and imaginary parts of G(jω)H(jω). Both are the function of ω.

iii) Equate imaginary part to zero to get equation as f(ω) = 0. Solve this to get value of ω which is making this imaginary part zero i.e. ω = ω_{pc}. **This frequency should be positive finite and greater than zero.** Otherwise it can be concluded that there is no intersection of polar plot with negative real axis.

iv) Substitute this value of ω_{pc} in the real part to get the actual co-ordinates of an intersection of point of polar plot with negative real axis. This point is denoted as Q.

4.13 : Steps to Sketch Nyquist Plot

Q.22 List the steps used to solve the problem using Nyquist criterion. [SPPU : May-16, Marks 6]

Ans. :

Step 1 : Count how many number of poles of G(s)H(s) are in the right half of s-plane i.e. with positive real part. This is the value of P.

Step 2 : Decide the stability criterion as N = – P i.e. how many times Nyquist plot should encircle '–1 +j0' point for absolute stability.

Step 3 : Select Nyquist path as per the function G(s)H(s).

Step 4 : Analyse the sections as starting point and terminating point of plot. Last section analysis is not required.

Step 5 : Mathematically find out ω_{pc} and intersection of Nyquist plot with negative real axis by rationalizing $G(j\omega)H(j\omega)$. This point of intersection is denoted as Q.

Step 6 : With the knowledge of step 4 and 5, sketch the Nyquist plot.

Step 7 : Count the number of encirclements N of –1 + j0 by Nyquist plot. If this matches with the criterion decided in step 2 system is stable, otherwise unstable.

$$G.M. = \frac{1}{|OQ|} \quad \text{where}$$

Q = Intersection point of Nyquist plot with negative real axis obtained in step 5.

$$G.M. = 20 \log_{10} \frac{1}{|OQ|} \text{ dB}$$

Q.23 For a certain control system $G(s)H(s) = \frac{K}{s(s+2)(s+10)}$. Sketch the Nyquist plot and hence calculate the range of values of K for stability. [SPPU : Dec.-02, 03, 11, May-06, Marks 8]

Ans. :

Step 1 : P = 0

Step 2 : N = – P = 0, the critical point –1 + j0 should not get encircled by Nyquist plot.

Step 3 : Pole at origin hence Nyquist path is as shown in the Fig. Q.23.1.

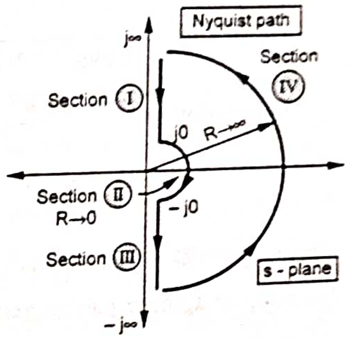


Fig. Q.23.1

Step 4 :

$$G(j\omega)H(j\omega) = \frac{K}{j\omega(2+j\omega)(10+j\omega)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{K}{\omega \times \sqrt{4+\omega^2} \times \sqrt{100+\omega^2}}$$

$$\phi = \frac{\tan^{-1}\left(\frac{0}{K}\right)}{\tan^{-1}\left(\frac{\omega}{0}\right)\tan^{-1}\left(\frac{\omega}{2}\right)\tan^{-1}\left(\frac{\omega}{10}\right)}$$

$$= -90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$

Section I : $s = +j\infty$ to $s = +j0$ i.e. $\omega \to \infty$ to $\omega \to +0$

Starting point	$\omega \to \infty$	$0 \angle -270^\circ$	$-90^\circ - (-270)^\circ = +180^\circ$ Anticlockwise rotation
Terminating point	$\omega \to +0$	$0 \angle +90^\circ$	

Section II : $s = +j0$ to $s = -j0$ i.e. $\omega \to +0$ to $\omega \to -0$

Starting point	$\omega \to +0$	$\infty \angle -90^\circ$	$90^\circ - (-90)^\circ = +180^\circ$ Anticlockwise rotation
Terminating point	$\omega \to -0$	$\infty \angle +90^\circ$	

Section III is mirror image of section about real axis.

Section IV is an origin.

Step 5 : $G(j\omega)H(j\omega) = \dfrac{K}{j\omega(10+j\omega)(2+j\omega)}$

Rationalizing,

$$G(j\omega)H(j\omega) = \frac{K(-j\omega)(10-j\omega)(2-j\omega)}{(j\omega)(-j\omega)(10+j\omega)(10-j\omega)(2+j\omega)(2-j\omega)}$$

$$\therefore \quad G(j\omega)H(j\omega) = \frac{-Kj\omega[20-12j\omega-\omega^2]}{\omega^2(4+\omega^2)(100+\omega^2)}$$

$$= \frac{-12K\omega^2}{D} - \frac{Kj\omega(20-\omega^2)}{D}$$

where $\quad D = \omega^2(4+\omega^2)(100+\omega^2)$

Equating imaginary part to zero,

$\omega(20-\omega^2) = 0 \quad$ i.e. $\quad \omega^2 = 20 \quad$ i.e. $\quad \omega_{pc} = \sqrt{20}$

Substituting in real part,

$$\text{Point Q} = \frac{-12K\times 20}{20\times(20+4)\times(100+20)} = -\frac{K}{240}$$

Step 6 : The Nyquist plot is,

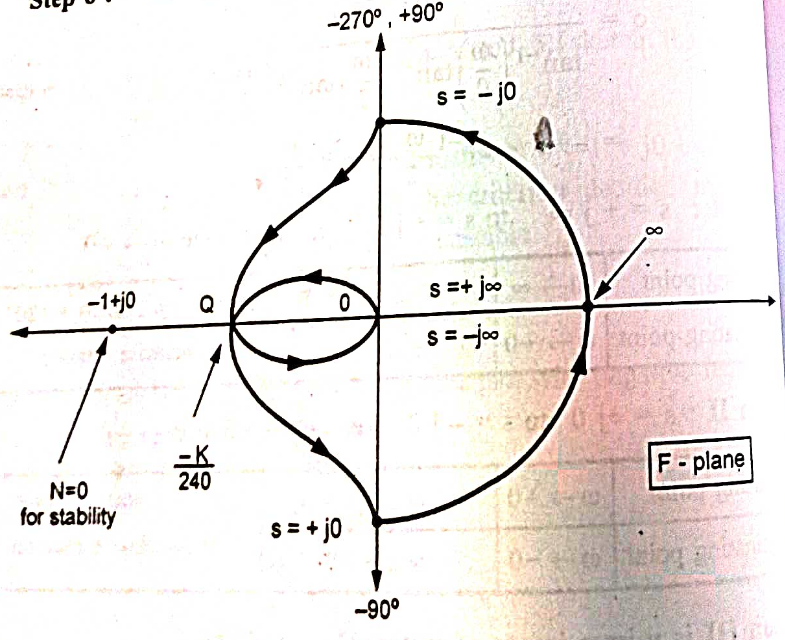


Fig. Q.23.2

Step 7 : Now for absolute stability, N = 0

i.e. it should be located on left side of point Q i.e. $|OQ| < 1$

$$\therefore \quad \left|-\frac{K}{240}\right| < 1$$

$$\therefore \quad K < 240$$

So range of values of K for stability is

$$0 < K < 240$$

Q.24 Construct Nyquist plot and find phase crossover frequency and gain margin if : $G(s)\cdot H(s) = \dfrac{1}{s(s+1)(s+2)}$. Also comment on stability. **[SPPU : Dec.-14, Marks 8]**

Ans. : Refer Q.23 for the procedure and verify the answer as :

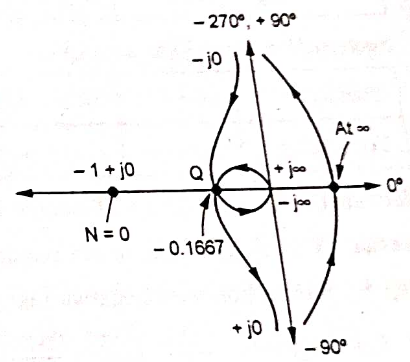
$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega)(2+j\omega)}$$

Rationalize to give,

$$G(j\omega)H(j\omega) = \frac{-3\omega^2 - j\omega(2-\omega^2)}{\omega^2(1+\omega^2)(4+\omega^2)}$$

$$\therefore \quad \omega_{pc} = \sqrt{2}, \quad Q = -0.1667$$

The Nyquist plot is shown in the Fig. Q.24.1.



$$G.M = 20\log\frac{1}{|OQ|}\ \text{dB}$$

$$= 20\log\frac{1}{0.1667} = +15.56\ \text{dB}$$

Fig. Q.24.1

Q.25 For the unity feedback system with open loop transfer function.

$$G(s) = \frac{20}{s(s+1)(s+10)},$$

sketch Nyquist plot and investigate stability. **[SPPU : May-17, Marks 8]**

Ans. : **Step 1 :** $P = 0$

Step 2 : $N = -P = 0$ for stability

Step 3 : The Nyquist plot is shown in the Fig. Q.25.1(a).

Step 4 : Analysis of sections,

$$G(j\omega)H(j\omega) = \frac{20}{j\omega(1+j\omega)(10+j\omega)}$$

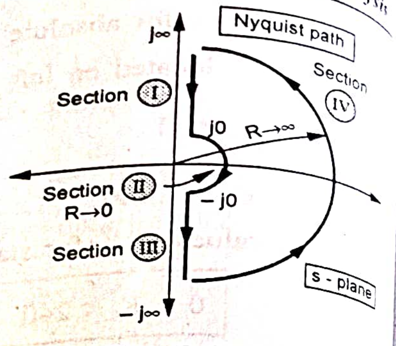


Fig. Q.25.1(a)

Section I : $s = +j\infty$ to $s = +j0$

Starting	$\omega \to \infty$	$0 \angle -270°$	$-90° - (-270°) = +180°$ Anticlockwise
Terminating	$\omega \to +0$	$\infty \angle -90°$	

Section II : $s = +j0$ to $s = -j0$

Starting	$\omega \to +0$	$\infty \angle -90°$	$+90° + 90° = +180°$ Anticlockwise
Terminating	$\omega \to -0$	$\infty \angle +90°$	

Section III is mirror image of section I about real axis.

Section IV is about origin so not required.

Step 5 : Intersection with negative real axis.

$$G(j\omega)H(j\omega) = \frac{20(-j\omega)(1-j\omega)(10-j\omega)}{j\omega(-j\omega)(1+j\omega)(1-j\omega)(10+j\omega)(10-j\omega)} \quad \text{...Rationalize}$$

$$= \frac{-220\omega^2 - 20j\omega(10-\omega^2)}{\omega^2(1+\omega^2)(100+\omega^2)}$$

Equating imaginary part to zero, $10 - \omega^2 = 0$

$\therefore \quad \omega_{pc} = \sqrt{10}$ rad/sec,

substitute in real part

$\therefore \quad$ Point $Q = \frac{-220}{11 \times 110} = -0.1818$

Step 6 : The Nyquist plot is shown in the Fig. Q.25.1 (b).

Step 7 : As $N = 0$, $-1 + j0$ is not encircled by Nyquist plot. This satisfies step 2, hence system is stable.

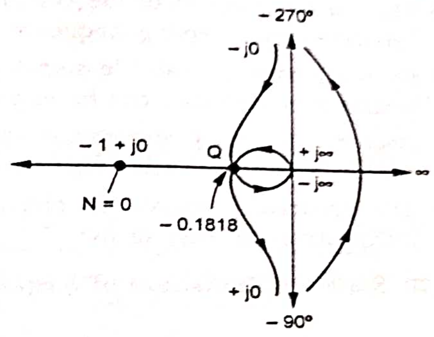


Fig. Q.25.1 (b)

4.14 : Advantages of Nyquist Plot

Q.26 State the advantages of Nyquist plot.

[SPPU : May-01, 03, Dec.-07, 09, Marks 4]

Ans. : 1) It gives same information about absolute stability as provided by Routh's criterion.

2) Useful for determining the stability of the closed loop system from open loop transfer function without knowing the roots of characteristic equation.

3) It also indicates relative stability giving the values of G.M. and P.M.

4) It indicates reality, the manner in which system should be compensated to yield desired response.

4.15 : Advantages and Disadvantages of Frequency Domain Analysis

Q.27 State the advantages of frequency domain analysis.

[SPPU : Dec.-22, Marks 8]

Ans. : 1) Without the knowledge of transfer function, frequency response can be obtained experimentally.

2) The methods are easy to use, calculations are simple and the designs are well tested.

3) The transfer function of the system can be determined practically in the laboratory by obtaining frequency response of the system.

4) By using readily available signal generators and the precise measuring instruments, accuracy can be improved.

5) Once the frequency response is known, the step response of the system can be predicted as there exists a close relation between the two.

6) The apparatus required to obtain the frequency response is simple, inexpensive and easy to use.

Q.28 State the limitations of frequency response analysis.

☞ [SPPU : May-22, Marks 8]

Ans. : 1) The method can be applied to only linear systems. For the systems with some degree of nonlinearity, the results are inaccurate and designs are not exact.

2) Due to digital computers and simulators, the frequency response methods are considered to be outdated and old.

3) Even for linear systems, the estimation of the step response from frequency response is obtained by using the fact that a higher order system behaves approximately as second order when underdamped. Hence the results are not that accurate. To have accurate results extensive calculations are required to be done.

4) For an existing system, the frequency response is possible only when its time constant is up to few minutes. If the time constant is in hours then practically the method is not convenient.

5) Obtaining frequency response practically is fairly time consuming.

END...

Unit V

5 State Space Representation

5.1 : Advantages of State Space Representation

Q.1 State the advantages of state space approach over transfer function approach.

☞[SPPU : May-06,08,14, Dec.-01,02,03,10,13,14,15, Marks 4]

Ans. : 1) The method takes into account the effect of all initial conditions.

2) It can be applied to non-linear as well as time varying systems.

3) It can be conveniently applied to multiple input multiple output systems.

4) The system can be designed for the optimal conditions precisely by using this modern method.

5) Any type of the input can be considered for designing the system.

6) As the method involves matrix algebra, can be conveniently adopted for the digital computers.

7) The state variables selected need not necessarily be the physical quantities of the system.

8) The vector matrix notation greatly simplifies the mathematical representation of the system.

Q.2 Compare the classical control theory with the state variable theory. ☞[SPPU : May-2000, 01, 07, Dec.-05, 07, 11, Marks 4]

Ans. :

S.N.	Classical control theory	State variable theory
1.	Initial conditions are nelected.	Effect of initial conditions is considered.
2.	Applied only to linear systems.	Applied to both linear and nonlinear systems.

3.	Difficult to apply to multiple input multiple output systems.	Easily applied to multiple input multiple output systems.
4.	Applied only to time invariant systems.	Applied to time variant and time invariant systems.
5.	Applied with very few special type of inputs.	Any type of input can be considered.
6.	Not suitable for digital computers.	Due to matrix algebra, easily adopted for ditital computers.
7.	It uses many trial and error procedures hence fails to give optimal solution.	Uses perfect mathematical procedures to give required optimal solution.

5.2 : Important Definitions

Q.3 Define the terms : i) State ii) State variables iii) State vector iv) State space **[SPPU : May-01, 04, 05, 07, 08, 10, 11, 22, Dec.-05, 06, 07, 09, 10, 13, Marks 4]**

Ans. : 1) State : The state of a dynamic system is defined as a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

2) State variables : The variables involved in determining the state of a dynamic system X(t), are called the state variables. $X_1(t), X_2(t), \ldots\ldots X_n(t)$ are nothing but the state variables. These are normally the energy storing elements contained in the system.

3) State vector : The 'n' state variables necessary to describe the complete behaviour of the system can be considered as 'n' components of a vector X(t) called the state vector at time 't'. The state vector X(t) is the vector sum of all the state variables.

4) State space : The space whose co-ordinate axes are nothing but the 'n' state variables with time as the implicit variable is called the state space.

5.3 : State Space Model of Linear Systems

Q.4 Explain the state model for multiple input multiple output control systems with the help of block diagram. **[SPPU : Dec.-05,06,07,09, May-04,05,07,08,10,11, Marks 6]**

Ans. : Consider multiple input multiple output, n^{th} order system as shown in the Fig. Q.4.1.

Number of inputs = m

Number of outputs = p

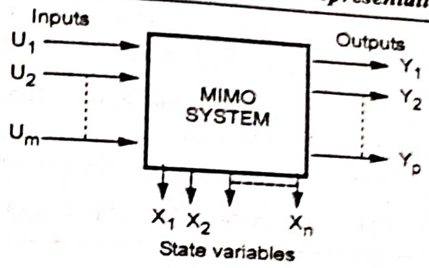


Fig. Q.4.1

Input, output and state variable vectors are column vectors having orders $m \times 1$, $n \times 1$ and $p \times 1$ respectively.

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ \vdots \\ U_m(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ \vdots \\ X_n(t) \end{bmatrix}, \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ \vdots \\ Y_p(t) \end{bmatrix}$$

- For such a system, the state variable representation can be arranged in the form of 'n' first order differential equations.

$$\frac{dX_1(t)}{dt} = \dot{X}_1(t) = f_1(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

$$\frac{dX_2(t)}{dt} = \dot{X}_2(t) = f_2(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

$$\vdots$$

$$\frac{dX_n(t)}{dt} = \dot{X}_n(t) = f_n(X_1, X_2, \ldots\ldots X_n, U_1, U_2 \ldots\ldots\ldots U_m)$$

- Integrating the above equation,

$$X_i(t) = X_i(t_0) + \int_{t_0}^{t} f_i(X_1, X_2 \ldots\ldots X_n \; U_1, U_2 \ldots\ldots U_m)\, dt$$

where i = 1, 2, ...n.

- Thus 'n' state variables and hence state vector at any time 't' can be determined uniquely.
- Any 'n' dimensional time invariant system has state equations in the functional form as, $\dot{X}(t) = f(X, U)$.
- While outputs of such system are dependent on the state of system and instantaneous inputs.

- Hence functional output equation can be written as, $Y(t) = g(X, U)$ where 'g' is the functional operator.
- For the linear time invariant systems, all the equations can be written in vector matrix form as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{and} \quad Y(t) = C\,X(t) + D\,U(t)$$

A = **System matrix** or **evolution matrix** of order $n \times n$

B = **Input matrix** or **control matrix** of order $n \times m$

C = **Output matrix** or **observation matrix** of order $p \times n$

D = **Direct transmission matrix** of order $p \times m$

- The two vector equations together is called the **state model** of the linear system.

5.4 : State Model using Physical Variables

Important Points to Remember

Selection of state variables

- To obtain the state model for a given system, it is necessary to select the state variables.
- **In general, the physical variables associated with energy storing elements, which are responsible for initial conditions, are selected as the state variables of the given system.**
- In electrical systems, the initial conditions are existing due to intitial current through inductors and initial voltage across the capacitors.
- Hence for the **electrical systems,** the **currents through** various **inductors** and the **voltage across the various capacitors** are selected to be the state variables.
- Then by any method of network analysis, the equations must be written interms of the selected state variables, their derivatives and the inputs. The equations must be rearranged in the standard form so as to obtain the required state model.
- **It is important that the equation for differentiation of one state variable should not involve the differentiation of any other state variable.**

- **In the mechanical systems, the displacements and velocities of energy storing elements such as spring and friction are selected as the state variables.**

Q.5 Obtain physical variable state model of the system shown in Fig. 7. **[SPPU : May -17, Marks 6]**

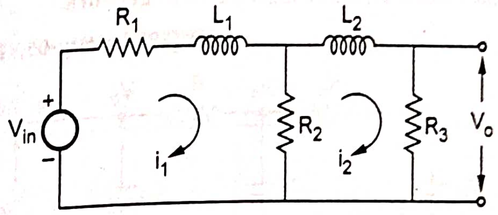


Fig. Q.5.1

Ans. : Applying KVL to the two loops,

$$-i_1R_1 - L_1\frac{di_1}{dt} - i_1R_2 + i_2R_2 + V_{in} = 0$$

$$\therefore \quad \frac{di_1}{dt} = \frac{-(R_1+R_2)}{L_1}i_1 + \frac{R_2}{L_1}i_2 + \frac{1}{L_1}V_{in} \quad \text{...(Q.5.1)}$$

$$-L_2\frac{di_2}{dt} - i_2R_3 - i_2R_2 + i_1R_1 = 0$$

$$\therefore \quad \frac{di_2}{dt} = +\frac{R_1}{L_2}i_1 - \frac{(R_2+R_3)}{L_2}i_2 \quad \text{...(Q.5.2)}$$

$$V_0 = i_2R_3 \quad \text{...(Q.5.3)}$$

Using $X_1 = i_1$, $X_2 = i_2$, $V_{in} = U$, $V_0 = Y$

$$\dot{X}_1 = \frac{-(R_1+R_2)}{L_1}X_1 + \frac{R_2}{L_1}X_2 + \frac{1}{L_1}U$$

$$\dot{X}_2 = \frac{R_1}{L_2}X_1 - \frac{(R_2+R_3)}{L_2}X_2$$

$$Y = X_2R_3$$

Hence state model is $\dot{X} = AX + BU$, $Y = CX + DU$ with,

$$A = \begin{bmatrix} \frac{-(R_1+R_2)}{L_1} & \frac{R_2}{L_1} \\ \frac{R_1}{L_2} & \frac{-(R_2+R_3)}{L_2} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \end{bmatrix}, C = [0 \quad R_3], D = [0]$$

Q.6 Find state model of following electrical network. [SPPU : May-05, 16, Marks 6]

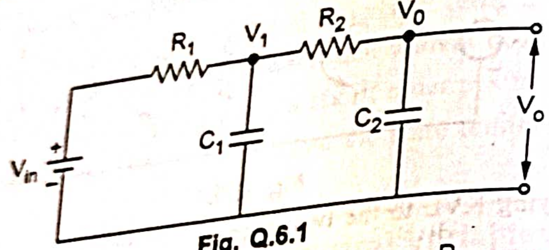


Fig. Q.6.1

Ans. : Assume the loop currents as shown in the Fig. Q.6.1 (a). Applying KVL to the loops,

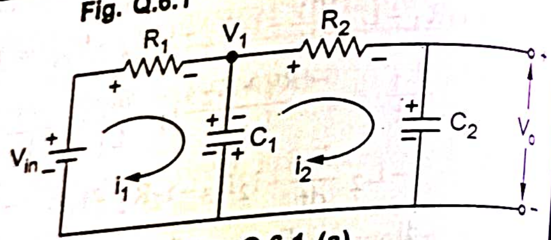


Fig. Q.6.1 (a)

$$-i_1 R_1 - V_1 + V_{in} = 0$$

$$\therefore i_1 = \frac{V_{in} - V_1}{R_1} \quad \text{... (Q.6.1)}$$

$$\therefore -i_2 R_2 - V_0 + V_1 = 0 \quad \text{i.e.} \quad i_2 = \frac{V_1 - V_0}{R_2} \quad \text{... (Q.6.2)}$$

V_1 is voltage across capacitor C_1 hence,

$$i_1 - i_2 = C_1 \frac{dV_1}{dt} \quad \text{and} \quad i_2 = C_2 \frac{dV_0}{dt}$$

Using (Q.6.1) and (Q.6.2) in above equations,

$$\frac{V_{in} - V_1}{R_1} - \left[\frac{V_1 - V_0}{R_2}\right] = C_1 \frac{dV_1}{dt} \text{ and } \frac{V_1 - V_0}{R_2} = C_2 \frac{dV_0}{dt}$$

Select the state variables as voltages across capacitors i.e. $X_1 = V_1$ and $X_2 = V_0$, $U = V_{in}$, $Y = V_0$

$$\frac{U}{R_1} - \frac{X_1}{R_1} - \frac{X_1}{R_2} + \frac{X_2}{R_2} = C_1 \dot{X}_1$$

$$\therefore \dot{X}_1 = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) X_1 + \frac{1}{C_1 R_2} X_2 + \frac{1}{C_1 R_1} U \quad \text{... (Q.6.3)}$$

$$\therefore \frac{X_1}{R_2} - \frac{X_2}{R_2} = C_2 \dot{X}_2 \quad \text{i.e.} \quad \dot{X}_2 = \frac{1}{C_2 R_2} X_1 - \frac{1}{C_2 R_2} X_2 \quad \text{... (Q.6.4)}$$

$$Y = V_0 = X_2 \quad \text{... (Q.6.5)}$$

So state model is $\dot{X} = AX + BU$, $Y = CX + DU$ with,

$$\therefore A = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{C_1 R_2} \\ \frac{1}{C_2 R_2} & \frac{1}{C_2 R_2} \end{bmatrix}, B = \begin{bmatrix} \frac{1}{C_1 R_1} \\ 0 \end{bmatrix} C = [0 \quad 1], D = [0]$$

5.5 : State Diagram Representation

Important Points to Remember

- State diagram is the pictorial representation of the state model derived for the given system.
- It is the proper interconnection of three basic unit : i) Scalars ii) Adders iii) Integrators
- Scalars are the multipliers, adders are summing points and integrators are the elements which integrate the differentiation of state variables to obtain required state variable.
- The transfer function of any integrator is always $\frac{1}{s}$.
- **The output of each integrator is always a state variable.**
- The Fig. 5.1 shows the state diagram of the basic state model $\dot{X}(t) = A X(t) + B U(t)$ and $Y(t) = C X(t) + D U(t)$.
- Practically state model is obtained from the state diagram by assuming the output of each integrator as a state variable.

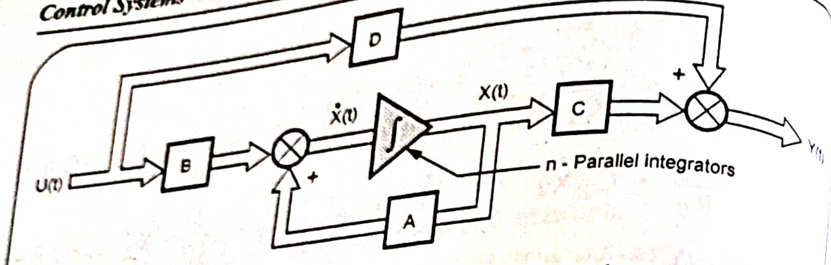


Fig. 5.1 State diagram of MIMO system

5.6 : State Model using Phase Variables

Q.7 Discuss the state space representation using phase variables.

Ans. : Consider a linear continuous time system represented by n^{th} order differential equation as,

$$Y^n + a_{n-1} Y^{n-1} + \dots + a_1 \dot{Y} + a_0 Y(t) = b_0 U(t) \quad \dots (Q.7.1)$$

Choice of state variable is generally output variable Y(t) itself. And other state variables are derivatives of the selected state variable Y(t).

$\therefore X_1(t) = Y(t),\quad X_2(t) = \dot{Y}(t) = \dot{X}_1(t),\quad X_3(t) = \ddot{Y}(t) = \ddot{X}_1(t) = \dot{X}_2(t) \dots\dots$

- Thus the various state equations are,

$$\dot{X}_1(t) = X_2(t),\ \dot{X}_2(t) = X_3(t),\ \dots\ \dot{X}_{n-1}(t) = X_n(t).$$

- Only n variables are to be defined to keep their number minimum.
- Thus $\dot{X}_{n-1}(t)$ gives n^{th} state variable $X_n(t)$. But to complete state model $\dot{X}_n(t)$ is necessary.
- $\dot{X}_n(t)$ is to be obtained by substituting the selected state variables in the original differential equation (Q.7.1).
- Use $Y(t) = X_1,\ \dot{Y}(t) = X_2,\ \ddot{Y}(t) = X_3,\ \ Y^{n-1}(t) = X_n(t),\ Y^n(t) = \dot{X}_n(t)$
- $\dot{X}_n(t) = -a_0X_1 - a_1X_2 \dots - a_{n-2} X_{n-1} - a_{n-1} X_n + b_0\ U(t) \quad \dots (Q.7.2)$
- Hence all the equations now can be expressed in vector matrix form as,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} U(t)$$

Thus $\dot{X} = A\ X(t) + B\ U(t)$.

- And as $Y(t) = X_1(t)$, the matrix $C = [1 \quad 0 \quad 0 \ \dots\ 0]$ and $D = [0]$. Such set of state variables is called set of **phase variables**. Such a form of matrix A is also called **Bush form** or **Companion form.**
- The various advantages of phase variables i.e. direct programming method are,

1. Easy to implement.
2. The phase variables need not be physical variables hence mathematically powerful to obtain state model.
3. It is easy to establish the link between the transfer function design and time domain design using phase variables.
4. In many simple cases, just by inspection, the matrices A, B, C and D can be obtained.

Q.8 Obtain the state model for system represented by

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 10y = 3U(t)$$

☞[SPPU : Dec.-07, Marks 10]

Ans. : System is 3rd order, n = 3

3 integrators and variables are required. Select $y = X_1$ and then successive differentiation of y as next variable.

$$\therefore \quad \dot{X}_1 = X_2 = dy/dt \quad \dots (Q.8.1)$$

$$\dot{X}_2 = X_3 = \frac{d^2y}{dt^2} \quad \dots (Q.8.2)$$

Now as 3 variables are defined, $\dot{X}_3 \neq X_4$ but $\dot{X}_3$ must be obtained by substituting all selected variables in original differential equation.

$$\therefore \quad \dot{X}_3 + 6X_3 + 11X_2 + 10X_1 = 3U \quad \text{as} \quad \dot{X}_3 = \frac{d^3y}{dt^3}$$

$$\dot{X}_3 = 3U - 10X_1 - 11X_2 - 6X_3$$

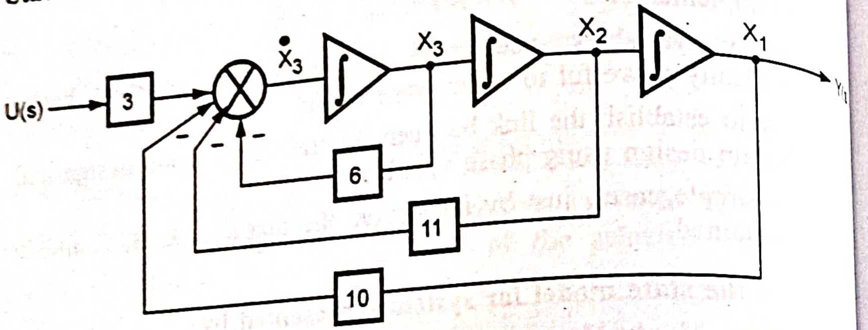
$\therefore$ $y = X_1$ which is output equation ... (Q.8)

$\therefore$ State model can be written as,

$$\dot{X} = AX + BU \text{ and } Y = CX$$

where
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -11 & -6 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix},\ C = [\,1\ \ 0\ \ 0\,]$$

State diagram :



5.7 : State Model using Direct Decomposition

Q.9 How state model is obtained by direct decomposition of the T.F. ?

Ans. : • Consider an element in T.F. as $\frac{1}{s+a}$. It can be written as,

$$\frac{1}{s+a} = \frac{\frac{1}{s}}{1+\frac{a}{s}} = \frac{G}{1+GH}$$

- Its simulation is as shown in the Fig. Q.9.1.
- If this group is added in the forward path of another such loop, we get block diagram as shown in the Fig. Q.9.2.

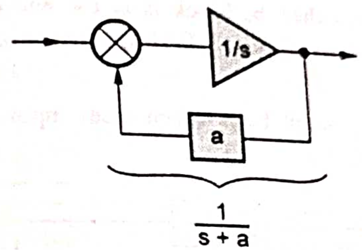


Fig. Q.9.1

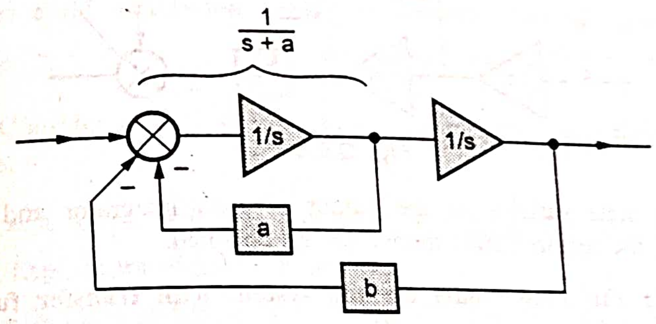


Fig. Q.9.2

$$\text{T.F.} = \frac{\frac{1}{s+a}\times\frac{1}{s}}{1+\frac{b}{s(s+a)}} = \frac{1}{sX+b} \qquad X = (s+a)$$

- If this entire group is placed in the forward path of another such loop with H = C than we get T.F. = $\frac{1}{sY+c}$ where $Y = sX+b,\ X = s+a$.
- Hence the denominators of various orders can be decomposed as,

$$s^2 + as + b \Rightarrow \{s(s+a)+b\}$$

$$s^3 + as^2 + bs + c \Rightarrow \{(s+a)s+b]s+c\}$$

$$s^4 + as^3 + bs^2 + cs + d \Rightarrow \{[s+a]+b\}s+c)s+d\}$$

- From this, the block diagram can be constructed.
- For numerator, if it is constant b_0 then add a block of b_0 at the end of block diagram obtained.

- If numerator is $b_0 + b_1 s$ then b_0 block is at the end of the block diagram while b_1 is taken from input of first integrator as shown in the Fig. Q.9.3.
- For $b_0 + b_1 s + b_2 s^2$, the block b_2 is taken from input of second integrator as shown in the Fig. Q.9.3.

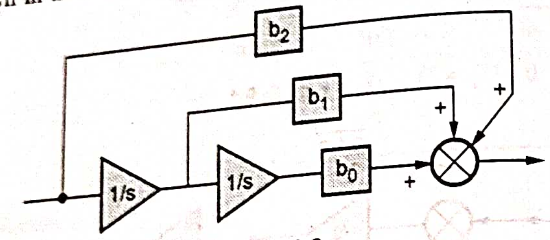


Fig. Q.9.3

- Assigning state variable as the output of each integrator and writing equations, the required state model can be obtained.

Q.10 Obtain the state model for the system with transfer function $\frac{Y(s)}{U(s)} = \frac{3s+4}{s^2+5s+6}$.

[SPPU : Dec.-22, Marks 9]

Ans. : The closed loop T.F. is given from which the denominator can be directly decomposed as,

$$s^2 + 5s + 6 = [(s + 5)\, s + 6]$$

The numerator is 3s + 4.

The state diagram is shown in the Fig. Q.10.1.

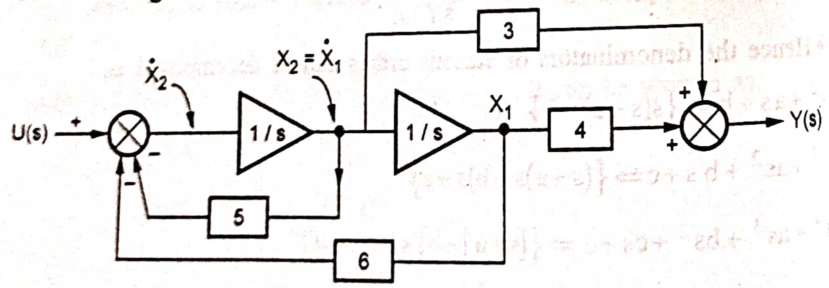


Fig. Q.10.1

From the state diagram,

$$\dot{X}_1 = X_2, \qquad \dot{X}_2 = -6X_1 - 5X_2 + U(s)$$

$$Y = 4X_1 + 3X_2$$

Hence $\dot{X} = AX + BU$ and $Y = CX + DU$ where

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [4 \quad 3],\ D = [0]$$

5.8 : Transfer Function from State Model

Q.11 Derive an expression to obtain transfer function from state model.

[SPPU : Dec.-11, May-10, 12, 14, Marks 4]

Ans. : Consider a standard state model derived for linear time invariant system as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \quad \text{and} \quad Y(t) = C\,X(t) + D\,U(t) \quad \ldots (Q.11.1)$$

- Taking Laplace transform of both sides,

$$[s\,X(s) - X(0)] = A\,X(s) + B\,U(s) \quad \text{and} \quad Y(s) = C\,X(s) + D\,U(s) \quad \ldots (Q.11.2)$$

- Note that as the system is time invariant, the coefficient of matrices A, B, C and D are constants. While the definition of transfer function is based on the assumption of zero initial conditions i.e. X(0) = 0.

$$\therefore \quad s\,X(s) = A\,X(s) + B\,U(s) \quad \text{i.e.} \quad s\,X(s) - A\,X(s) = B\,U(s)$$

- As 's' is an operator while A is matrix of order n × n hence to match the orders of two terms on left hand side, multiply 's' by identity matrix I of the order n × n.

$$\therefore \quad sI\,X(s) - A\,X(s) = B\,U(s) \quad \text{i.e.} \quad [sI - A]\,X(s) = B\,U(s)$$

- Premultiplying both sides by $[sI - A]^{-1}$,

$$[sI - A]^{-1}\,[sI - A]\,X(s) = [sI - A]^{-1}\,B\,U(s)$$

But $[sI - A]^{-1}\,[sI - A] = 1$

$$\therefore \quad X(s) = [sI - A]^{-1}\,B\,U(s) \quad \ldots (Q.11.3)$$

- Substituting in the equation (Q.10.2),

$Y(s) = C\,[sI - A]^{-1} B\,U(s) + D\,U(s) = \{C\,[sI - A]^{-1} B + D\}\,U(s)$

- Hence the transfer function is,

$$T(s) = \frac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B + D \quad \ldots (Q.11.4)$$

- Now $[sI - A]^{-1} = \frac{Adj\,[sI-A]}{|sI-A|}$ hence,

$$\therefore \quad T(s) = \frac{C\,Adj\,[sI-A]\,B}{|sI-A|} + D \quad \ldots (Q.11.5)$$

Q.12 Consider a system having state model

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} U \text{ and } Y = [\,1 \;\; 1\,] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

with D = 0. Obtain its T.F. ☞ **[SPPU : May-12,16, Dec.-13,19, Marks 8]**

Ans. : $\quad T.F. = C\,[sI - A]^{-1}\,B$

$$[sI - A]^{-1} = \frac{Adj\,[sI - A]}{|sI - A|}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ +4 & 2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s+2 & 3 \\ -4 & s-2 \end{bmatrix}$$

$$Adj = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s-2 & 4 \\ -3 & s+2 \end{bmatrix}^T = \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$$

$$|sI - A| = (s+2)(s-2) + 12 = s^2 - 4 + 12 = s^2 + 8$$

$$[sI - A]^{-1} = \frac{\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}}{s^2 + 8}$$

$$\therefore \quad T.F. = \frac{[1 \;\; 1] \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}}{s^2 + 8} = [\,1 \;\; 1\,] \frac{\begin{bmatrix} 3s - 21 \\ 5s + 22 \end{bmatrix}}{s^2 + 8}$$

$$= \frac{[8s + 1]}{s^2 + 8}$$

Q.13 Find transfer function of $= \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} r(t)$ **;**

$y = [1 \;\; 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

☞ **[SPPU : May-22, Marks 9]**

Ans. : From the given model,

$$A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix},\; B = \begin{bmatrix} 2 \\ 5 \end{bmatrix},\; C = [1 \;\; 2]$$

$$T.F. = C[sI - A]^{-1} B = \frac{C\,Adj[sI - A]\,B}{|sI - A|}$$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}$$

$$Adj\,[sI - A] = \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}$$

$$|sI - A| = (s+5)(s+1) + 3$$
$$= s^2 + 6s + 8$$
$$= (s+2)(s+4)$$

$$\therefore \quad T.F. = \frac{[1 \;\; 2] \begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}}{(s+1)(s+4)} = \frac{[1 \;\; 2] \begin{bmatrix} 2s-3 \\ 5s+31 \end{bmatrix}}{(s+2)(s+4)}$$

$$\therefore \quad T.F. = \frac{12s + 59}{(s+2)(s+4)}$$

Q.14 Derive the formula for obtaining transfer function from state model and use it to find transfer function of a system with state model.

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -4 & -7 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$y = [2 \quad 3]\, X$ **[SPPU : Dec.-17, Marks 6]**

Ans. : Refer Q.11 for derivation of T.F.

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -7 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [2 \quad 3] \quad \ldots D = 0$$

$$T(s) = \frac{C\,Adj\,[sI-A]\,B}{|sI-A|}$$

$$[sI-A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -4 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 4 & s+7 \end{bmatrix}$$

$$Adj\,[sI-A] = \begin{bmatrix} s+7 & 1 \\ -4 & s \end{bmatrix}$$

$$|sI-A| = s^2+7s+4$$

$$\therefore \quad T(s) = \frac{[2 \quad 3]\begin{bmatrix} s+7 & 1 \\ -4 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s^2+7s+4} = \boxed{\frac{3s+2}{s^2+7s+4}}$$

Q.15 Determine the transfer function of system with state model :

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$y = [1 \quad 2 \quad 1]\, x$ **[SPPU : Dec.-18, Marks 6]**

Ans. : From the given model,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -7 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [1 \quad 2 \quad 1]$$

$$T(s) = \frac{Y(s)}{U(s)} = C\,[sI-A]^{-1}\,B + D, \quad D = [0]$$

$$[sI-A]^{-1} = \frac{Adj\,[sI-A]}{|sI-A|}$$

$$[sI-A] = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -7 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 4 & s+7 \end{bmatrix}$$

$$\text{Cofactor}\,[sI-A] = \begin{bmatrix} s^2+7s+4 & -3 & -3s \\ s+7 & s(s+7) & -4s-3 \\ 1 & s & s^2 \end{bmatrix}$$

$$\therefore \quad Adj\,[sI-A] = \{\text{Cofactor}\,[sI-A]\}^T$$

$$= \begin{bmatrix} s^2+7s+4 & s+7 & 1 \\ -3 & s(s+7) & s \\ -3s & -4s-3 & s^2 \end{bmatrix}$$

$$|sI-A| = s^3+7s^2+3+4s = s^3+7s^2+4s+3$$

$$\therefore \quad T.F. = \frac{C\,Adj\,[sI-A]\,B}{|sI-A|}$$

$$= \frac{[1 \quad 2 \quad 1]\begin{bmatrix} s^2+7s+4 & s+7 & 1 \\ -3 & s(s+7) & s \\ -3s & -4s-3 & s^2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{s^3+7s^2+4s+3}$$

$$= \frac{[1 \quad 2 \quad 1]\begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix}}{s^3+7s^2+4s+3} = \boxed{\frac{s^2+2s+1}{s^3+7s^2+4s+3}}$$

5.9 : State Transition Matrix and its Properties

Q.16 Write a note on state transition matrix and its properties.

[SPPU : May-01,08,13,16,17,18,19 Dec.-05,06,09,10,12,13,15,16,22, Marks 7]

Ans. : Consider a scalar differential equation as,

$$\frac{dx}{dt} = ax \quad \text{where} \quad x(0) = x_0 \qquad \dots (Q.16.1)$$

- This is a homogeneous equation without the input vector.
- The required solution of such a homogeneous equation in scalar form is,

$x(t) = e^{at} x_0$

- Thus if the homogeneous state equation is considered, $\dot{X}(t) = A\,X(t)$ then its solution can be written as,

$$\boxed{X(t) = e^{At}\,X(0)}$$

- In this case, e^{At} **is not a scalar but a matrix of order** $n \times n$ as that of **matrix A.**
- It can be observed that without input, initial state X(0) drives the state X(t) at any time t. Thus there is transition of the initial state X(0) from initial time t = 0 to any time t through the matrix e^{At}. Hence e^{At} is called **state transition matrix** denoted as $\phi(t)$.
- The various useful properties of the state transition matrix are,

$$\phi(t) = e^{At} = \text{State transition matrix}$$

1. $\phi(0) = e^{A \times 0} = I = \text{Identity matrix}$
2. $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$ i.e. $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1)\,\phi(t_2) = \phi(t_2)\,\phi(t_1)$
4. $e^{A(t+s)} = e^{At}\,e^{As}$
5. $e^{(A+B)t} = e^{At}\,e^{Bt}$ only if AB = BA
6. $[\phi(t)]^n = [e^{At}]^n = e^{Ant} = \phi(nt)$

7. $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = \phi(t_2 - t_0)$

- This property states that the process of transition of state can be divided into number of sequential transition. Thus t_0 to t_2 can be divided as t_0 to t_1 and t_1 to t_2, as stated in the property.

8. $\phi(t)$ is a **non-singular matrix** for all finite values of t.

5.10 : State Transition Matrix by Laplace Transform Method

Q.17 How to obtain state transition matrix by Laplace transform method ? [SPPU : Dec.-16,22, May-17,18,19, Marks 4]

Ans. : Consider the non-homogeneous state equation as,

$$\dot{X}(t) = A\,X(t) + B\,U(t) \qquad \dots (Q.17.1)$$

Taking Laplace transform of both sides,

$$s\,X(s) - X(0) = A\,X(s) + B\,U(s)$$

$$\therefore \quad s\,X(s) - A\,X(s) = X(0) + B\,U(s)$$

As s is operator, multiplying it by Identify matrix of order $n \times n$,

$$[sI - A]\,X(s) = X(0) + B\,U(s)$$

Premultiplying both sides by $[sI - A]^{-1}$,

$$\therefore [sI - A]^{-1}\,[sI - A]\,X(s) = [sI - A]^{-1}\,\{X(0) + B\,U(s)\}$$

$$\therefore \quad X(s) = [sI - A]^{-1}\,X(0) + [sI - A]^{-1}\,B\,U(s) \qquad \dots (Q.17.2)$$

$$= ZIR + ZSR$$

The zero input response is given by,

$$X(t) = e^{At}X(0) \quad \text{i.e.} \quad X(s) = \phi(s)X(0)$$

Comparing two expressions of X(s) we can write,

$\phi(s) = L[e^{At}] = [sI-A]^{-1}$ = State transition matrix.

$$\therefore \quad e^{At} = L^{-1}[\phi(s)] = L^{-1}[sI-A]^{-1} = L^{-1}\left\{\frac{\text{Adj }[sI-A]}{|sI-A|}\right\}$$

Important Points to Remember

- Consider a square (n × n) matrix A and the element a_{ij}. If now i^{th} row and j^{th} column are deleted then the remaining (n – 1) rows and columns form a determinant M_{ij}. The value of this determinant is called the minor of an element a_{ij}. Then cofactor C_{ij} of an element a_{ij} is defined as,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

- Adjoint of a matrix is the transpose of co-factor matrix.

$$\therefore \quad \text{Adj } A = [\text{Cofactor matrix of } A]^T$$

- For 2 × 2 matrix, the adjoint can be obtained directly by interchanging the diagonal elements and changing the sign of remaining elements.

Q.18 Determine the state transition matrix of state equation

$$X = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix} x(t).$$

[SPPU : May-22, Marks 9]

Ans. :

$$A = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix}$$

$$[sI-A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+9 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{\text{Adj}[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+9 & 1 \\ -8 & s \end{bmatrix}}{s^2+9s+8}$$

$$= \begin{bmatrix} \frac{s+9}{(s+1)(s+8)} & \frac{1}{(s+1)(s+8)} \\ \frac{-8}{(s+1)(s+8)} & \frac{s}{(s+1)(s+8)} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1.1428}{s+1} - \frac{0.1428}{s+8}\right) & \left(\frac{0.1428}{s+1} - \frac{0.1428}{s+8}\right) \\ \left(\frac{-1.1428}{s+1} + \frac{1.1428}{s+8}\right) & \left(\frac{-0.1428}{s+1} + \frac{1.1428}{s+8}\right) \end{bmatrix}$$

$$\therefore \quad e^{At} = L^{-1}[sI-A]^{-1}$$

$$= \begin{bmatrix} 1.1428e^{-t} - 0.1428e^{-8t} & 0.1428e^{-t} - 0.1428e^{-8t} \\ -1.1428e^{-t} + 1.1428e^{-8t} & -0.1428e^{-t} + 1.1428e^{-8t} \end{bmatrix}$$

Q.19 Determine the state transition matrix of system with state equation.

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} X$$

[SPPU : Dec.-17,19, May-18, Marks 6]

Ans. : $e^{At} = L^{-1}[sI-A]^{-1}$

$$[sI-A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+6 \end{bmatrix}$$

$$\text{Adj }[sI-A] = \begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix},$$

$$|sI-A| = s^2 + 6s + 8 = (s+2)(s+4)$$

$$\therefore \quad e^{At} = L^{-1}\left\{\frac{\text{Adj }[sI-A]}{[sI-A]}\right\} = L^{-1}\left\{\frac{\begin{bmatrix} s+6 & 1 \\ -8 & s \end{bmatrix}}{(s+2)(s+4)}\right\}$$

$$= L^{-1}\begin{bmatrix} \frac{s+6}{(s+2)(s+4)} & \frac{1}{(s+2)(s+4)} \\ \frac{-8}{(s+2)(s+4)} & \frac{s}{(s+2)(s+4)} \end{bmatrix} = L^{-1}\begin{bmatrix} \frac{2}{s+2} - \frac{1}{s+4} & \frac{0.5}{s+2} - \frac{0.5}{s+4} \\ \frac{-4}{s+2} + \frac{4}{s+4} & \frac{-1}{s+2} + \frac{2}{s+4} \end{bmatrix}$$

$$\therefore \quad e^{At} = \begin{bmatrix} 2e^{-2t} - e^{-4t} & 0.5\,e^{-2t} - 0.5\,e^{-4t} \\ -4e^{-2t} + 4e^{-4t} & -e^{-2t} + 2\,e^{-4t} \end{bmatrix}$$

5.11 : Solution of Homogeneous State Equation

Important Points to Remember

- The solution of nonhomogeneous state equation is divided into two parts called : Zero input response (ZIR) and Zero state response (ZSR).
- By Laplace transform method it is given by,

$$X(t) = L^{-1}[\phi(s)]\,X(0) + L^{-1}[\phi(s)\,B\,U(s)]$$

$$= e^{At}\,X(0) + L^{-1}[\phi(s)\,B\,U(s)] \quad = ZIR + ZSR$$

where $\phi(s) = [sI - A]^{-1} = \dfrac{\text{Adj}\,[sI-A]}{|sI-A|}$

Q.20 Find out the time response for unit step input of a system given by $\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ 5 \end{bmatrix} U(t)$ **and** $Y(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t)$ **and** $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

[SPPU : May-07,19, Dec.-12, Marks 8]

Ans. : $[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|sI - A| = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\therefore \quad e^{At} = L^{-1}\,[(sI - A)^{-1}]$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} = \phi(s)$$

Finding partial fractions,

$$e^{At} = L^{-1}\,(\phi(s)) = \begin{bmatrix} \dfrac{2}{s+1} - \dfrac{1}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ \dfrac{-2}{s+1} + \dfrac{2}{s+2} & \dfrac{-1}{s+1} + \dfrac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$ZIR = e^{At}\,X(0) = e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

To find $ZSR = L^{-1}\,\{\phi(s) B U(s)\}$

$U(t)$ = Unit step $\therefore U(s) = 1/s$

$$\therefore \quad ZSR = L^{-1} \left\{ \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} [1/s] \right\}$$

$$= L^{-1} \left\{ \begin{matrix} \dfrac{5}{s(s+1)(s+2)} \\ \dfrac{5}{(s+1)(s+2)} \end{matrix} \right\} = L^{-1} \left\{ \begin{matrix} \dfrac{2.5}{s} - \dfrac{5}{s+1} + \dfrac{2.5}{s+2} \\ \dfrac{5}{s+1} - \dfrac{5}{s+2} \end{matrix} \right\}$$

$$= \begin{bmatrix} 2.5 - 5e^{-t} + 2.5\,e^{-2t} \\ 5\,e^{-t} - 5e^{-2t} \end{bmatrix}$$

$$\therefore \quad X(t) = ZIR + ZSR = \begin{bmatrix} 2.5 - 3e^{-t} + 1.5\,e^{-2t} \\ 3\,e^{-t} - 3e^{-2t} \end{bmatrix}$$

$$\therefore \quad Y(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} X(t)$$

$$\therefore \quad \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 2.5 - 3e^{-t} + 1.5e^{-2t} \\ 3e^{-t} - 3e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} - 3e^{-2t} \\ -5 + 6e^{-2t} - 3e^{-t} \end{bmatrix}$$

$$\therefore \quad Y_1(t) = 3(e^{-t} - e^{-2t}) \text{ and } Y_2(t) = -5 + 3(2e^{-2t} - e^{-t})$$

These are the outputs for unit step input applied.

Q.21 Determine state transition matrix of the system with state equation :

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} x .$$

Also determine solution of state equation if :

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

☞ **[SPPU : Dec.-18, Marks 6]**

Ans. : From the model, $A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$

$$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+4 \end{bmatrix}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s+4 & 1 \\ 0 & s \end{bmatrix}, \quad |sI - A| = s(s+4)$$

$$\therefore \quad e^{At} = L^{-1}\left\{\frac{\text{Adj}\,[sI-A]}{|sI-A|}\right\} = L^{-1}\begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+4)} \\ 0 & \frac{1}{s+4} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{s} & \frac{0.25}{s} - \frac{0.25}{s+4} \\ 0 & \frac{1}{s} \end{bmatrix}$$

$$\therefore \quad \boxed{e^{At} = \begin{bmatrix} 1 & 0.25 - 0.25e^{-4t} \\ 0 & 1 \end{bmatrix}}$$

The solution is given by,

$$X(t) = e^{At}\, X(0) = \begin{bmatrix} 1 & 0.25 - 0.25e^{-4t} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad \boxed{X(t) = \begin{bmatrix} 1.25 - 0.25e^{-4t} \\ 1 \end{bmatrix}}$$

5.12 : Controllability and Observability

Q.22 What is controllability and observability ?

☞ **[SPPU : May-11, Dec.-12, Marks 6]**

Ans. : • The concept of controllability of a system is related to the transfer of any initial state of the system to any other desired state, in a finite length of time by application of proper inputs.

- A system is said to be **completely state controllable** if it is possible to transfer the system state from any initial state $X(t_0)$ to any other desired state $X(t_f)$ in a specified finite time interval (t_f) by a control vector U(t).
- According to Kalman's test, the **necessary and sufficient condition** for the system to be completely state controllable is that **the rank of the composite matrix Q_c is 'n'** where matrix Q_c is given by,

$$\boxed{Q_c = [B : AB : A^2B : \ldots A^{n-1}B]} \quad \ldots (Q.22.1)$$

- The observability is related to the problem of determining the system state by measuring the output for finite length of time.
- A system is said to be **completely observable**, if every state $X(t_0)$ can be completely identified by measurements of the outputs Y(t) over a finite time interval. If the system is not completely observable means that few of its state variables are not practically measurable and are shielded from the observation.
- According to Kalman's test, the system is completely observable if and only if the rank of the composite matrix Q_o is 'n' where Q_o is given by,

$$\boxed{Q_o = [C^T : A^T C^T : \ldots\ldots\ldots (A^T)^{n-1} C^T]}$$

Q.23 Investigate for complete state controllability and state observability of system with state space model matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = [1 \;\; 0 \;\; 2]$$

☞ [SPPU : May -17, Marks 6]

Ans. : For controllability,

$$Q_c = [B : AB : A^2B], \; n = 3$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \quad \ldots A^2B = A[AB]$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}, \; |Q_c| = 1 \neq 0$$

Thus rank of $Q_c = 3 = n$ hence system is **completely state controllable.**

For observability,

$$Q_o = [C^T \;\; A^TC^T \;\; (A^T)^2C^T]$$

$$A^TC^T = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -10 \\ -1 \\ -4 \end{bmatrix}$$

$$(A^T)^2C^T = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} -10 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 20 \\ -6 \\ +7 \end{bmatrix}$$

$$\ldots (A^T)^2C^T = A^T(A^TC^T)$$

$$\therefore \quad Q_o = \begin{bmatrix} 1 & -10 & 20 \\ 0 & -1 & -6 \\ 2 & -4 & 7 \end{bmatrix}$$

$$|Q_o| = 129 \neq 0$$

Thus rank of $Q_o = 3 = n$ hence system is completely state observable.

Q.24 Investigate for complete state controllability and state observability of system with state space model matrices :

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -5 & -7 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = [1 \;\; 2 \;\; 1]$$

☞ [SPPU : Dec.-17, Marks 7]

Ans. : Refer Q.23 for the procedure and verify the answers as,

$$Q_c = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -7 \\ 1 & -7 & 44 \end{bmatrix}, \quad |Q_c| = 1 \neq 0,$$

rank = n = 3 hence system is **controllable.**

$$Q_o = \begin{bmatrix} 1 & -4 & 20 \\ 2 & -4 & 21 \\ 1 & -5 & 31 \end{bmatrix}, \quad |Q_o| = +25 \neq 0,$$

rank = n = 3 hence system is **observable.**

Q.25 Investigate for complete state controllability and observability of the system with state model :

$$\dot{x} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u$$

$$y = [0 \;\; 0 \;\; 1] \, x$$

☞ [SPPU : May-18, Marks 7]

Ans. : For controllability,

$$Q_c = [B : AB : A^2B]$$

$$AB = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A^2B = A[AB] = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \\ -1 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 1 & 0 & -6 \\ 2 & 1 & -8 \\ 0 & 2 & -1 \end{bmatrix}, \; |Q_c| = -9 \neq 0$$

Thus, the rank of $Q_c = n = 3$. Hence system is **completely controllable.**

For observability, $Q_o = [C^T : A^T C^T : (A^T)^2 C^T]$

$$A^T C^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -3 \end{bmatrix}, \; |Q_o| = 1 \neq 0$$

Thus, the rank of $Q_o = n = 3$. Hence, system is **completely observable.**

Q.26 For the system with state model :

$$\dot{x} = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \;\; 0 \;\; 0]\, x$$

investigate the state controllability and state observability.

☞[SPPU : Dec.-18, Marks 7]

Ans. : From given model,

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, C = [1 \;\; 0 \;\; 0]$$

For controllability, $Q_c = [B : AB : A^2B]$

$$AB = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$A^2B = A[AB] = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 4 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ 2 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 0 & 14 \\ 0 & 4 & 2 \end{bmatrix}, \; |Q_c| = +12 \neq 0$$

As $|Q_c| \neq 0$, the rank of Q_c is $n = 3$ hence the system is completely **controllable.**

For observability, $Q_o = [C^T : A^T C^T : (A^T)^2 C^T]$

$$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T C^T = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T (A^T C^T) = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 4 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\therefore \quad Q_o = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 6 \end{bmatrix}, \; |Q_o| = 8 \neq 0$$

As $|Q_o| \neq 0$, the rank of Q_o is $n = 3$ hence the system is completely **observable.**

Q.27 Investigate for complete state controllability and observability of the system with state model :

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1, -1]$$

☞[SPPU : May-19, Marks 7]

Ans. : For controllability, $Q_c = [B \; AB]$

$$AB = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\therefore \quad Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \; |Q_c| = -1 \neq 0 \text{ hence } r = 2 = n$$

Thus the system is **completely controllable.**

For observability, $Q_o = [C^T \; A^T C^T]$, $C^T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A^T C^T = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$\therefore$ $Q_o = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}$, $|Q_o| = 3 - 3 = 0$ hence $r = 1 \neq n$

$\therefore$ Hence the system is not completely observable.

Q.28 Investigate the complete state controllability and observability of the system with state model :

$$\dot{X} = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} u$$

$Y = [0 \;\; 0 \;\; 1] X$

[SPPU : Dec.-19, Marks 7]

Ans. : From given model,

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad C = [0 \;\; 0 \;\; 1]$$

$$C^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For controllability, $Q_c = [B : AB : A^2 B]$

$$AB = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A^2 B = A[AB] = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ $Q_c = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $|Q_c| = 0$ i.e. rank $r \neq n \neq 3$

Hence system is not completely controllable.

For observability, $Q_o = [C^T : A^T C^T : (A^T)^2 C^T]$

$$A^T C^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$(A^T)^2 C^T = A^T [A^T C^T] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -3 & -4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $Q_o = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, $|Q_o| = 0$ i.e. rank $r \neq n \neq 3$

Hence system is not completely observable.

Q.29 Find controllability and observability of the state model :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = [1 \;\; 1 \;\; 1], \quad D = [0].$$

[SPPU : May-15, Dec.-19, Marks 7]

Ans. : Refer Q.23 for the procedure and verify the answer that the rank of both the matrices Q_c and Q_o is not $n = 3$ and hence the system is **not completely controllable** and **not completely observable.**

5.13 : Controllable and Observable Canonical Forms

Q.30 With the help of general equation, explain concept of controllable canonical and observable canonical form of state space.

[SPPU : May-14, 15, Marks 7]

Ans. : Consider the transfer function of the system as,

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

- The **controllable cononical form** is nothing but phase variable form and can be obtained by direct decomposition of transfer function.

- Thus for the given transfer function, it can be expressed as,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_{n-1} \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = [b_n - a_n b_0 \vdots b_{n-1} - a_{n-1} b_0 \vdots \cdots \vdots b_1 - a_1 b_0] \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + [b_0] U$$

- While the **observable canonical form** can be obtained from controllable canonical form with A as the transpose of A and exchanging matrices B and C.
- Thus the observable canonical form is given by,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_{n-1} \\ \dot{X}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \vdots & & \cdots & & \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_1 - a_1 b_0 \end{bmatrix} U$$

Q.31 Obtain a state space representation in controllable and observable canonical form for the system $G(s) = \frac{s+3}{s^2+3s+2}$.

☞ **[SPPU : Dec.-15, May-22, Marks 6]**

Ans. : Controllable canonical form is phase variable form obtained by direct decomposition.

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{\{(s+3)s+2\}}$$

The state diagram is shown in the Fig. Q.31.1.

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = -2X_1 - 3X_2 + U$$

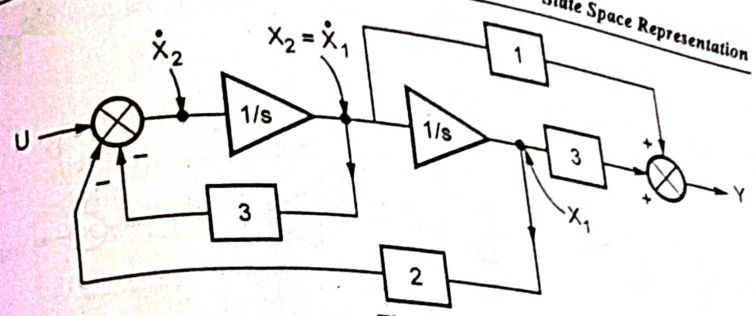


Fig. Q.31.1

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$Y = [3 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ as } Y = 3X_1 + X_2$$

$$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [3 \quad 1]$$

From this model observable canonical form can be obtained as,

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} U, \quad Y = [0 \quad 1] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad C = [0 \quad 1]$$

Q.32 Obtain controllable canonical and observable canonical state models for the system with transfer function :

$$G(s) = \frac{s^2+3s+5}{s^3+5s^2+2s+9}.$$

☞ **[SPPU : May-17,19, Marks 6]**

Ans. : The controllable canonical form is phase variable form. So using direct decomposition,

$$G(s) = \frac{s^2 + 3s + 5}{\{[(s+5)s+2]s+9\}}$$

The state diagram is shown in the Fig. Q.32.1.

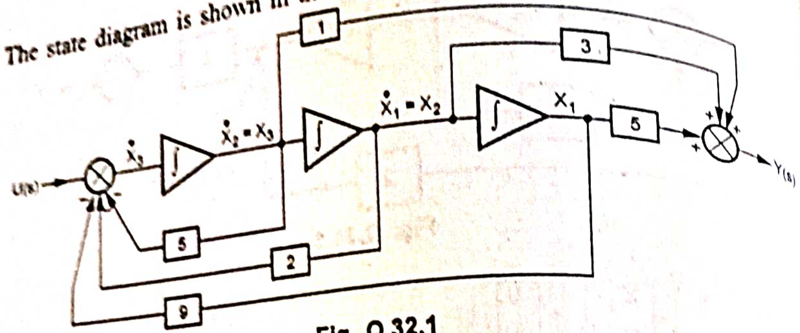


Fig. Q.32.1

$$\dot{X}_1 = X_2, \dot{X}_2 = X_3, \dot{X}_3 = -9X_1 - 2X_2 - 5X_3 + U$$

$\therefore$ $Y = 5X_1 + 3X_2 + X_3$

Hence the state model is $\dot{X} = AX + BU$ and $Y = CX + DU$ with,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -2 & -5 \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [5 \quad 3 \quad 1], D = [0]$$

This is controllable canonical form.

The observable canonical form is with,

$$A = \begin{bmatrix} 0 & 0 & -9 \\ 1 & 0 & -2 \\ 0 & 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, C = [0 \quad 0 \quad 1], D = [0]$$

Q.33 Obtain controllable canonical and observable canonical state **models for the system with transfer function :**

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 2s + 3}.$$

[SPPU : Dec.-17, Marks 7]

Ans. : Using direct decomposition,

$$G(s) = \frac{s^2 + 7s + 2}{\{[(s+9)s+2]s+3\}}$$

The state diagram is shown in the Fig. Q.33.1,

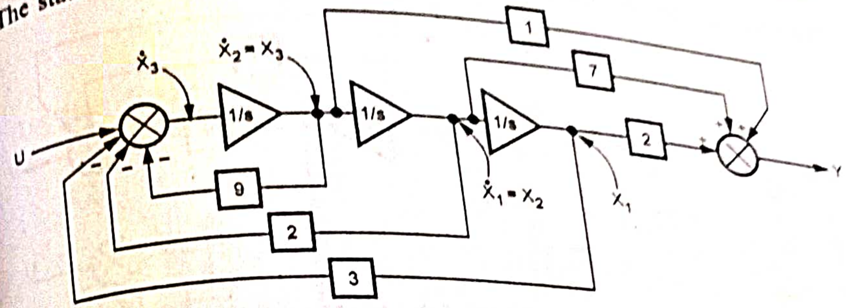


Fig. Q.33.1

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3,$$

$\therefore$ $\dot{X}_3 = -3X_1 - 2X_2 - 9X_3 + U$

$Y = 2X_1 + 7X_2 + X_3$

$\therefore$ $\dot{X} = AX + BU$ and $Y = CX + DU$ with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -2 & -9 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [2 \quad 7 \quad 1], D = [0]$$

This is controllable canonical form.

Form this observable canonical state model is given by the matrices,

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 1 & -9 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 7 \\ 1 \end{bmatrix}, C = [0 \quad 0 \quad 1], D = [0]$$

Q.34 Obtain the controllable canonical and observable canonical state models for the system with transfer function :

$$G(s) = \frac{s^2 + s + 9}{s^3 + 4s^2 + 11s + 3}$$

[SPPU : May-18, Marks 6]

Ans. : Using direct decomposition,

$$G(s) = \frac{s^2 + s + 9}{\{[(s+4)s+11]s+3\}}$$

The state diagram is shown in the Fig. Q.34.1.

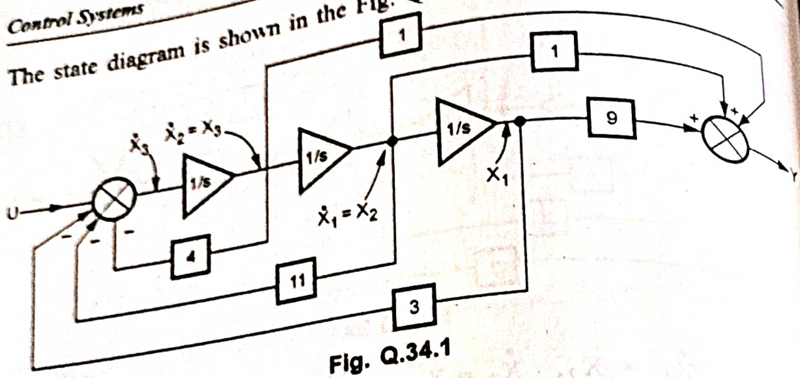


Fig. Q.34.1

From the state diagram,

$$\dot{X}_1 = X_2, \dot{X}_2 = X_3,$$

$$\dot{X}_3 = -3X_1 - 11X_2 - 4X_3 + U$$

$$Y = 9X_1 + X_2 + X_3$$

Hence, the canonical controllable state model is with,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -11 & -4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [9 \ \ 1 \ \ 1]$$

The observable canonical state model is with,

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 1 & 0 & -11 \\ 0 & 1 & -4 \end{bmatrix}, B = \begin{bmatrix} 9 \\ 1 \\ 1 \end{bmatrix}, C = [0 \ \ 0 \ \ 1]$$

Q.35 Obtain controllable canonical and observable canonical state model of the system with transfer function :

$$G(s) = \frac{s^2 + 7s + 9}{s^3 + 6s^2 + 4s + 3}$$

[SPPU : Dec.-18, Marks 7]

Ans. : Using direct decomposition,

$$G(s) = \frac{s^2 + 7s + 9}{\{[(s+6)s + 4]s + 3\}}$$

The state diagram is shown in the Fig. Q.35.1.

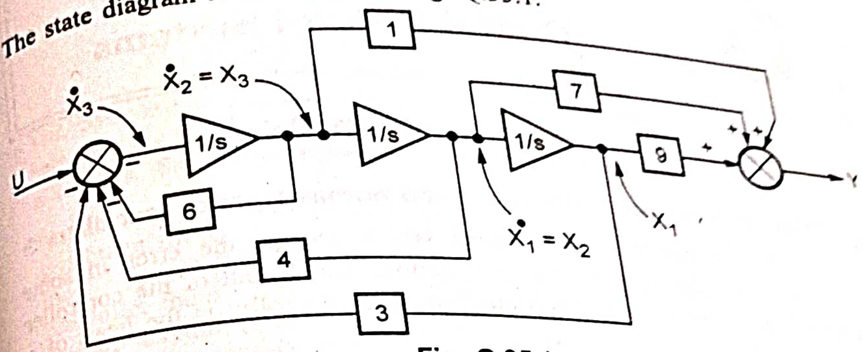


Fig. Q.35.1

$$\therefore \dot{X}_1 = X_2, \ \dot{X}_2 = X_3, \ \dot{X}_3 = -3X_1 - 4X_2 - 6X_3 + U$$

$$Y = 9X_1 + 7X_2 + X_3$$

$$\therefore \dot{X} = AX + BU \text{ and } Y = CX \text{ with,}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -6 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = [9 \ \ 7 \ \ 1], D = [0]$$

This is controllable canonical form.

From this, observable canonical state model is given by,

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 1 & 0 & -4 \\ 0 & 1 & -6 \end{bmatrix}, B = \begin{bmatrix} 9 \\ 7 \\ 1 \end{bmatrix}; C = [0 \ \ 0 \ \ 1], D = [0]$$

ENI

Unit VI

6 Controllers and Digital Control Systems

6.1 : Concept of Controller

Q.1 What is controller ? Which are the various types of controllers ?

Ans. : The controller is an element which accepts the error in some form and decide the proper corrective action. The output of the controller is applied to the process to be controlled. The controller is the heart of a control system.

- The various types of controller are, on-off controller, proportional (P) type, derivative (D) type, integral (I) type, proportional+integral (PI) type, proportional+derivative (PD) type and proportional+integral+derivative (PID) type of controller.

6.2 : On-Off Controller and Dead Zone

Q.2 Explain basic on-off controller and concept of dead zone. Also state its advantages and disadvantages.

Ans. : ON-OFF controller has to control two positions of control element, either on or off. Hence this mode is also called ON-OFF controller mode.

- This controller mode has two possible output states namely 0 % or 100 %. Mathematically this can be expressed as,

$$p = 0\,\%, \quad e_p < 0 \quad \text{and} \quad p = 100\,\%, \quad e_p > 0$$

- The p is the controller output and e_p is error based on the percent of span.
- Schematically it is reperesented as shown in the Fig.Q.2.1.
- Thus if the error rises above a certain critical value, the output changes from 0 % to 100 %. If the error decreases below certain critical value, the output falls from 100 % to 0 %.

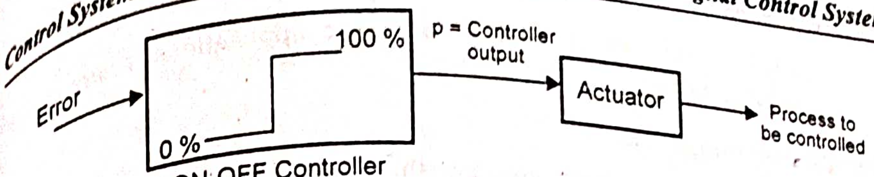


Fig. Q.2.1 Block diagram of ON-OFF controller

- The best example is a room heater. If the temperature drops below a setpoint, the heater is turned ON and if the temperature increases above a setpoint, the heater is turned OFF.
- In all the practical implementations of the ON-OFF controller there is an overlap as the error increases through zero or decreases through zero. Such an overlap creates a span of error in which there is no change in the controller output. This span is called **neutral zone, dead zone** or **dead band.** This is shown in the Fig. Q.2.2.

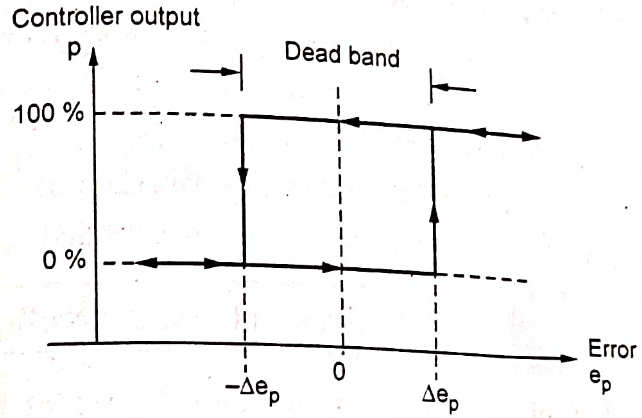


Fig. Q.2.2 ON-OFF controller with dead band

- It can be seen that till the error changes by Δe_p there is no change in the controller output. Similarly while decreasing also the error must decrease beyond Δe_p below 0 to change the controller output. Hence during the range of $2\Delta e_p$, there is no change in the controller output. This zone is also called the differential gap.
- In such a controller, the control variable always oscillates with a frequency which increases with decreasing width of the dead band. Hence dead band is purposely designed to prevent the oscillations in ON-OFF controllers.
- The **advantages** of ON-OFF controller are,

  i) It is cheapest of all the controllers.

  ii) It is simple to design and having least complexity.

iii) Useful in industry as well as domestic applications where rough control is required.

- **The disadvantages are,**
  i) Precise control of controlled variable is not possible.
  ii) The controlled variable oscillates about the final steady state value.
  iii) Due to presence of undershoot and overshoot the transient conditions are not much improved.
- **The applications are,**
  i) Preferred for large scale systems with relatively slow process rate.
  ii) In room air conditioners.
  iii) ON-OFF control of a heater
  iv) Liquid level control in large volume tank
  v) Temperatrue control in various applications.

6.3 : Proportional Controller (P)

Q.3 Explain the proportional controller stating its characteristics.

☞ [SPPU : Dec.-05,09,17,22, May-06,09,15,16,17,18,22, Marks 4]

Ans. : Proportional controller (P) : In this control mode, the output of the controller is simple proportional to the error e(t).

- The relation between the error e(t) and the controller output p is determined by constant called **proportional gain constant** denoted as K_p. The output of the controller is a linear function of the error e(t).
- Though there exists linear relation between controller output and the error, for a zero error the controller output should not be zero, otherwise the process will come to halt. Hence there exists some controller output p_o for the zero error.
- Hence mathematically the proportional control mode is expressed as,

$$p(t) = K_p e(t) + p_o \quad \text{... (Q.3.1)}$$

where K_p = Proportional gain constant.

p_o = Controller output with zero error.

Characteristics :

1. When the error is zero, the controller output is constant equal to p_o.
2. If the error occurs, then for every 1 % of error the correction of K_p % is achieved. If error is positive, K_p % correction gets added to p_o and if error is negative, K_p% correction gets subtracted from p_o.
3. The band of error exists for which the output of the controller is between 0 % to 100 % without saturation.
4. The gain K_p and the error band PB are inversely proportional to each other.
5. It produces an **offset error** in the output.

6.4 : Integral Controller (I)

Q.4 Explain the integral controller stating its characteristics.

☞ [SPPU : Dec.-05, 09, 22, May-06, 10, 18, 22, Marks 4]

Ans. : • The controller which is based on the history of error is called integral controller.

- In such a controller, the value of the controller output p(t) is changed at a rate which is proportional to the actuating error signal e(t).
- Mathematically it is expressed as,

$$\frac{d\,p(t)}{dt} = K_i e(t)$$

where K_i = Constant relating error and rate

- The constant K_i is also called integral constant. Integrating the above equation, actual controller output at any time t can be obtained as,

$$p(t) = K_i \int e(t)dt + p(0) \quad \text{... (Q.4.1)}$$

where p(0) = Controller output when integral action starts i.e. at t = 0.

- The output signal from the controller, at any instant is the area under the actuating error signal curve up to that instant. If the value of the error is doubled, the value of p(t) varies twice as fast i.e. rate of the controller output change also doubles.
- If the error is zero, the controller output is not changed.

Characteristics

1. If error is zero, the output remains at a fixed value equal to what is was, when the error became zero.
2. If the error is not zero, then the output begins to increase or decrease, at a rate K_i % per second for every ± 1 % of error.

- In pure integral mode, error can oscillate about zero and can be cyclic. Hence in practice integral mode is never used alone but combined with the proportional mode, to enjoy the advantages of both the modes.

6.5 : Derivative Controller (D)

Q.5 Explain the derivative controller stating its characteristics.

[SPPU : Dec.-05, 09, 22, May-06, 12, 18, 22, Marks 4]

Ans. : • In this mode, the output of the controller depends on the time rate of change of the actual errors. Hence it is also called **rate action mode** or **anticipatory action mode.**

- The mathematical equation for the mode is,

$$p(t) = K_d \frac{d\,e(t)}{dt}$$

where K_d = Derivative gain constant.

- The derivative gain constant indicates by how much % the controller output must change for every % per sec rate of change of the error. Generally K_d is expressed in minutes.
- The important feature of this type of control mode is that for a given rate of change of error signal, there is a unique value of the controller output.
- The advantage of the derivative control action is that it responds to the rate of change of error and can produce the significant correction before the magnitude of the actuating error becomes too large.
- Derivative control thus anticipates the actuating error, initiates an early corrective action and tends to increase stability of the system improving the transient response.

Characteristics

1. For a given rate of change of error signal, there is a unique value of the controller output.

2. When the error is zero, the controller output is zero.
3. When the error is constant i.e. rate of change of error is zero, the controller output is zero.
4. When error is changing, the controller output changes by K_d % for even 1 % per second rate of change of error.

6.6 : Composite Controllers

Important Points to Remember

- To take the advantages of various modes together, the composite control modes are used. The various composite control modes are,
  1. Proportional + Integral mode (PI)
  2. Proportional + Derivative mode (PD)
  3. Proportional + Integral + Derivative mode (PID)

6.7 : Poroportional + Integral Controller (PI)

Q.6 Explain the features of PI controller.

[SPPU : May-15, 16, 17, Dec.-15, 17, 18, Marks 4]

Ans. : • This is a composite control mode obtained by combining the proportional mode and the integral mode.

- The mathematical expression for such a composite control is,

$$p(t) = K_p\, e(t) + K_p K_i \int_0^t e(t)dt + p(0)$$

where $p(0)$ = Initial value of the output at $t = 0$

- The important advantage of this control is that one to one correspondence of proportional mode is available while the offset gets eliminated due to integral mode.

Characteristics :

1. When the error is zero, the controller output is fixed at the value that integral mode had when the error went to zero. This is nothing but p(0).

2. When the error is not zero, proportional mode adds the correction while the integral term starts increasing or decreasing from its initial value depending upon reverse or direct action.
3. It improves the steady state accuracy.
4. It increases the rise time so response becomes slow.
5. It decreases bandwidth of the system.
6. It filters out the high frequency noise.
7. It makes the response more oscillatory.

- PI mode can be used in the systems with the frequent or large load changes.
- The Fig. Q.6.1 shows the block diagram of PI controller.

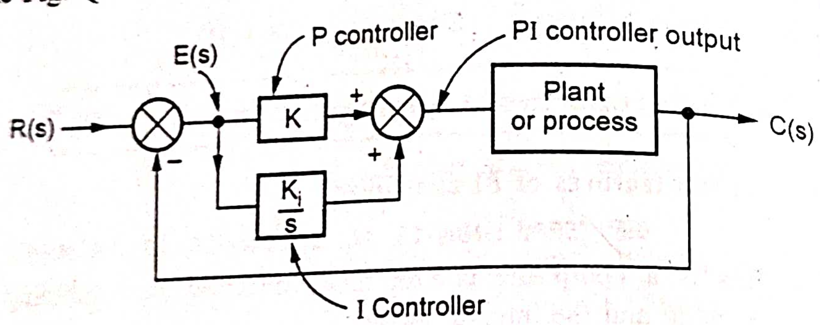


Fig. Q.6.1 PI controller

6.8 : Proportional + Derivative Controller (PD)

Q.7 Explain the features of PD controller.

[SPPU : May-10, 12, Dec.-05, 09, 15, 18, Marks 4]

Ans. : The series combination of proportional and derivative control modes gives proportional plus derivative control mode.

- The mathematical expression for the PD composite control is,

$$p(t) = K_p e(t) + K_p K_d \frac{d\,e(t)}{dt} + p(0)$$

Characteristics :

1. It improves the damping and reduces overshoot.
2. It reduces the rise time and makes response fast.
3. It makes the response stable very fast.
4. It improves the bandwidth of the system.
5. It can not eliminate offset error.
6. It may make the noise dominant at high frequencies.
7. It is not very effective for lightly damped systems.
8. It may require a relatively large capacitor while the circuit implementation.

The Fig. Q.7.1 shows the block diagram of PD controller.

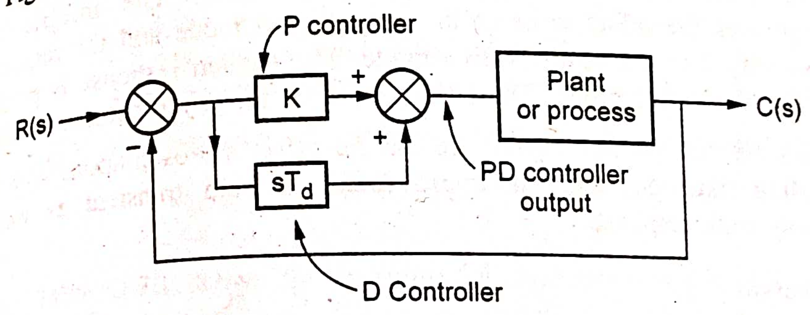


Fig. Q.7.1 PD controller

6.9 : PID Controller

Q.8 Write a note on PID controller and role of each action in short.

[SPPU : May-06, 11, 12, 13, 14, 16, Dec.-05, 07, 09, 10, 11, 12, 13, 15, 16, 17, 19, Marks 5]

Ans. : The composite controller including the combination of the proportional, integral and derivative control mode is called PID control mode and the controller is called three mode controller.

- It is very much complex to design but very powerful in action.
- Mathematically such a control mode can be expressed as,

$$p(t) = K_p\, e(t) + K_p K_i \int_0^t e(t)dt + K_p K_d \frac{d\,e(t)}{dt} + p(0)$$

where p(0) = Initial value of the output

- The Fig. Q.8.1 shows the block diagram of PID controller.

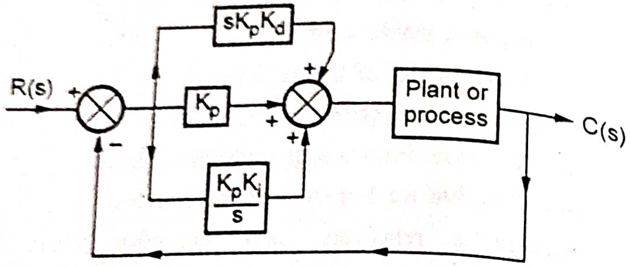


Fig. Q.8.1

- This mode has advantages of all the modes. The integral mode eliminates the offset error of the proportional mode and the response is also very fast due to derivative mode. The sudden response is produced due to derivative mode. Thus it can be used for any process condition.
- With the PID control action, there is no offset, no oscillations with least settling time. So there is improvement in both transient as well as steady state response.

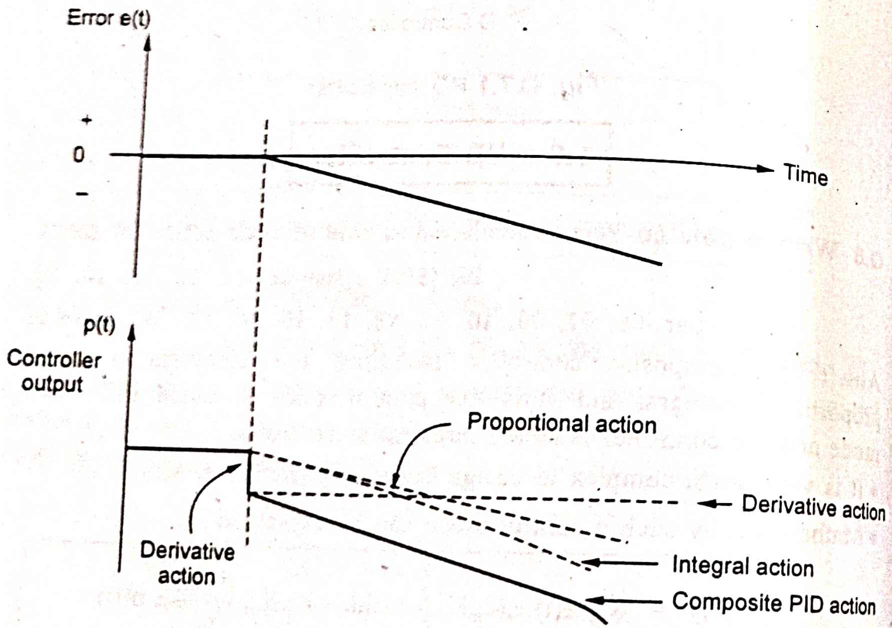


Fig. Q.8.2

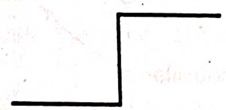
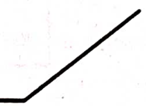
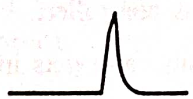
- The Fig. Q.8.2 shows the response of PID control for a particular error signal, assuming direct action.
- The proportional and PD control produces the offset error. It requires significant time to attain the steady state.
- The PI control eliminates the offset but at the expense of higher maximum overshoot, a long period of oscillations and more settling time.
- The PD control produces the steady state very quickly with least oscillations and smallest maximum overshoot but offset is significant.
- With PID control, there is no offset and system achieves the steady state with less settling time. Thus PID is the ultimate process composite controller.

6.10 : Step Response of Controllers

Q.9 Sketch the responses of various controllers for the step type of input. **[SPPU : May-15, 17, Dec.-14, Marks 6]**

Ans. :

- The Table Q.9.1 shows the response of various control modes to unit step load change.

Step input	
Response of P controller	
Response of I controller	
Response of D controller	

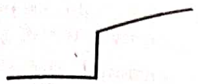
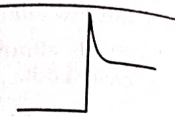
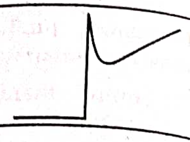
Response of PI controller	
Response of PD controller	
Response of PID controller	

Table Q.9.1

6.11 : Rate Feedback Controller

Q.10 Explain the rate feedback controller.

Ans. : • This controller uses the feedback which is derivative of output signal internally and compares with signal proportional to the error.

• It is also called output derivative controller.

• The Fig. Q.10.1 shows the use of rate feedback controller with standard second order system.

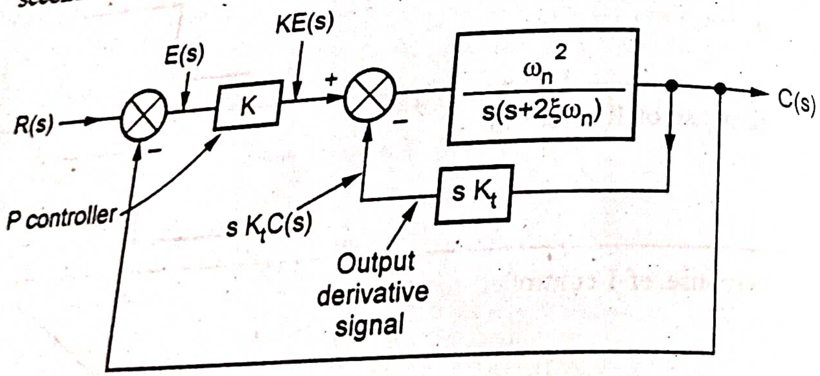


Fig. Q.10.1 Rate feedback controller

Due to this controller,

1. Damping ratio gets improved.
2. Reduces settling time and rise time.

3. It affects the steady state.
4. The natural frequency is unaffected.
5. It makes the response fast.

• But due to added zero, it is necessary to consider stability of the system while designing this controller.

6.12 : Integral Reset

Q.11 Explain the integral reset in PID controller. [SPPU : May-22, Marks 6]

Ans. : When the system with PID controller has constant error, the difference between the set point and the process variable never reaches to zero. In such a case the integral term can grow to vary large value. The integral term accumulates the signed error remaining after each control cycle to use it in the next control cycle. When the output of a controller becomes limited and process variable is not at its set point with constant error, the integral remainder term continues to increase. This is called **integral reset.** For example a control valve, when it is fully open or fully closed and the process variable is not at its set point then the integral remainder grows to very large value. When the process condition changes , integral reset becomes so large that even when the sign of the error changes, the output may not respond until all the integral reset remainder term is used up. This may cause excess overshooting.

Such an integral reset problem can be avoided by,

1. Increasing the set point in a suitable ramp.
2. Disabling the integral action till the process variable enters in to controlloable region.
3. Preventing the integral term to accumulate the error above or below the predetermined limits.
4. To make the integral value zero every time the error is equal to zero or crosses zero.

The integral reset occurs because of limitations of a physical system compared to an ideal system. The ideal output is physically impossible. The practical output is limited between the specific upper and lower bounds and due to which the error becomes constant. This is very common in position of a control valve which can be maximum fully opened or minimum fully closed and thus has upper and lower limits on its output. When this happens, the error becomes constant and integral reset occurs. During this time the controller output can not affect the controlled process variable. This is similar to the saturation condition.

Now a days external reset feedback is used to avoid the integral reset.

6.13 : Zeigler-Nicholas Method

Q.12 What is tuning of a controller ? Explain Zeigler-Nicholas method for tuning a PID controller.

[SPPU : May-18,22, Dec.-22, Marks 6]

Ans. : Finding suitable values of the constants K_p, T_i and K_d which makes the performance of a PID controller stable, optimum, robust and fast is called tuning of a controller.

Zeigler-Nicholas method uses following steps to tune the controller :

Step 1 : Bring the process as close as the specified set point manually.

Step 2 : Turn the PID controller in P mode with $T_i = \infty$ and $K_d = 0$. Increase K_p such that overall closed loop is in a continuous oscillations.

Step 3 : The value of K_p for which system shows sustained oscillations is called critical or ultimate gain denoted as K_u. The time between two successive peaks in the continuously oscillating output is called ultimate time period of oscillations denoted as T_u.

Step 4 : The table is provided by Ziegler-Nichols which gives results to design the various constants of controllers based on the values of K_u and T_u. Thus depending upon the type of controller P, PI or PID, using the values in the Table Q.12.1, the controller can be tuned.

Controller type	K_p	T_i	K_d
P	$0.5 K_u$	∞	0
PI	$0.45 K_u$	$\frac{T_u}{1.2} = 0.833\ T_u$	0
PID	$0.6 K_u$	$\frac{T_u}{2} = 0.5\ T_u$	$\frac{T_u}{8} = 0.125\ T_u$

Table Q.12.1

Hence the T.F. of PID controller is,

$$G_c(s) = K_p + \frac{K_p}{T_i}\frac{1}{s} + K_p K_d s$$

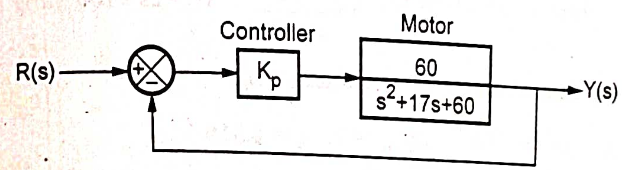
Using the Table Q.12.1, the final $G_c(s)$ is,

$$\therefore \quad G_c(s) = \frac{0.075 K_u T_u \left(s + \frac{4}{T_u}\right)^2}{s}$$

6.14 : Solved Examples on Controllers

Q.13 The system given below is so design of have damping ratio 0.707. Determine the required values of K_p for the given damping ratio.

[SPPU : Dec.-16, Marks 8]



Ans. : $\xi = 0.707$

The closed loop T.F. with K_p is,

$$\frac{Y(s)}{R(s)} = \frac{\frac{60\,K_p}{s^2+17s+60}}{1+\frac{60\,K_p}{s^2+17s+60}} = \frac{60\,K_p}{s^2+17s+60(K_p+1)}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$$\omega_n^2 = 60(K_p+1) \quad \text{i.e.} \quad \omega_n = \sqrt{60(K_p+1)}$$

$$2\xi\omega_n = 17 \quad \text{i.e.} \quad \xi = \frac{17}{2\omega_n}$$

$$\therefore \quad 0.707 = \frac{17}{2\sqrt{60(K_p+1)}}$$

$\therefore \sqrt{60(K_p+1)} = 12.0226$ ... Squaring and solving.

$K_p = 1.409$

Q.14 A unity feedback system has the plant transfer function $G(s) = \frac{C(s)}{M(s)} = \frac{10}{s(s+2)}$. A proportional plus derivative control is employed to control the dynamics of the system. Determine

i) The damping factor and undamped natural frequency when $K_d = 0$

ii) The value of K_d such that damping factor is 0.6.

☞ **[SPPU : May-17, Marks 8]**

Ans. : i) When $K_d = 0$ then,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{10}{s(s+2)}}{1+\frac{10}{s(s+2)}} = \frac{10}{s^2+2s+10}$$

Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$,

$\omega_n^2 = 10, \quad \omega_n = \sqrt{10}, \quad 2\xi\omega_n = 2, \quad \xi = \mathbf{0.316}$

ii) With K_d, system is as shown in the Fig. Q.14.1.

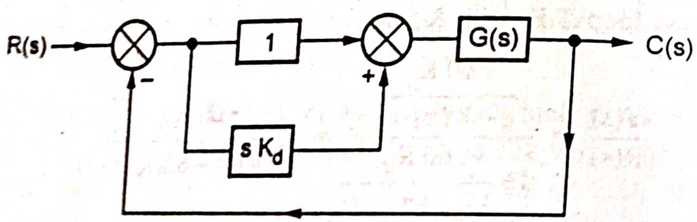


Fig. Q.14.1

$$\therefore \quad G(s) = \frac{(1+sK_d)10}{s(s+2)}, \quad H(s) = 1$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{\frac{(1+sK_d)10}{s(s+2)}}{1+\frac{(1+sK_d)10}{s(s+2)}} = \frac{10(1+sK_d)}{s^2+s(2+10K_d)+10}$$

$\therefore \quad \omega_n^2 = 10, \quad \omega_n = \sqrt{10}, \quad 2\xi\omega_n = 2+10K_d$

$$\xi = \frac{2+10K_d}{2\sqrt{10}} \quad \text{but} \quad \xi = 0.6 \text{ (given)}$$

$$\therefore \quad 0.6 = \frac{2+10K_d}{2\sqrt{10}} \quad \text{i.e.} \quad K_d = \mathbf{0.1794}$$

Q.15 Design a PID controller for system with unity feedback and $G(s) = \frac{K}{(s+3)(s^2+s+1)}$.

☞ **[SPPU : May-16, Marks 8]**

Ans.: Let K is the proportional controller gain. The characteristic equation is

$1 + G(s)H(s) = 0$

$$\therefore 1+\frac{K}{(s+3)(s^2+s+1)} = 0 \quad \text{i.e.} \quad s^3+4s^2+4s+K+3 = 0$$

Routh's array is,

s^3	1	4
s^2	4	K + 3
s^1	$\frac{13-K}{4}$	0
s^0	K + 3	

For critical value of K,

$13 - K = 0$

$\therefore K_u = K = \mathbf{13}$

$A(s) = 4s^2 + K + 3 = 0$

$\therefore s^2 = \frac{-(K+3)}{4} = -4$

$\therefore s = \pm j2$

$\therefore$ Frequency of sustained oscillations = 2 rad/s

But $\omega = 2\pi f = \frac{2\pi}{T_u}$ i.e. $T_u = \frac{2\pi}{2} = 3.1416$

Hence according to Ziegler - Nichols method,

$K_p = 0.6\,K_u = 0.6 \times 13 = \mathbf{7.8}$, $T_i = 0.5\,T_u = \mathbf{1.5708}$

$K_d = \frac{T_u}{8} = \mathbf{0.3927}$

Hence the T.F. of the PID controller is,

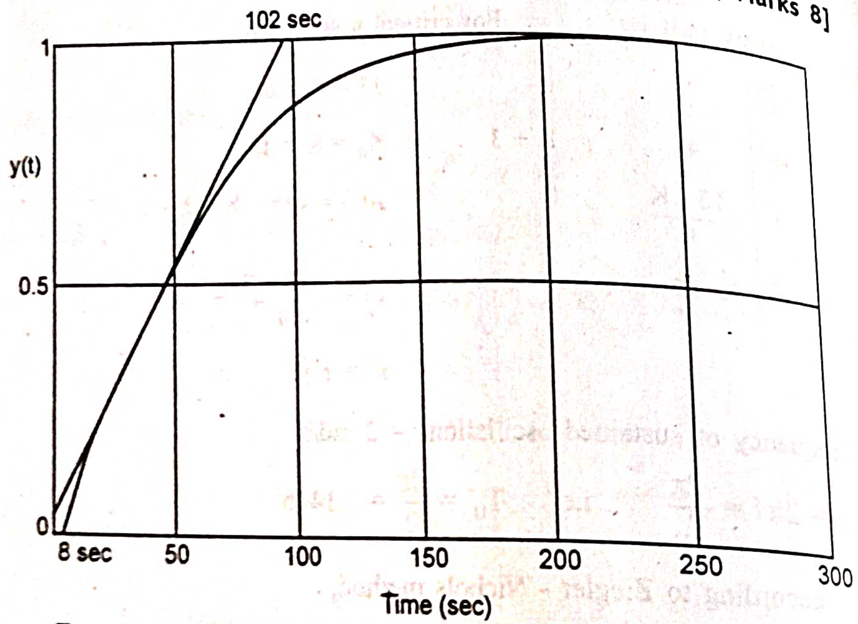
$$\therefore G_c(s) = \frac{0.075 K_u T_u \left(s + \frac{4}{T_u}\right)^2}{s} = \boxed{\frac{3.063\,(s+1.2732)^2}{s}}$$

Q.16 An open loop test of a temperature control system yields the reaction curve shown below. The system open loop transfer function is given by

$$G(s) = \frac{1}{(20s+1)(50s+1)}$$

Used Ziegler Nichols method to determine K_p, K_i, K_d ? For a quarter step response PID control system.

☞ **[SPPU : Dec.-16, Marks 8]**



Ans. : From the given quarter step response.

$$L = 8, \quad T = 102 - 8 = 94$$

For PID control

$$K_p = \frac{1.2\,T}{L} = \frac{1.2 \times 94}{8} = 14.1$$

$$T_i = 2L = 16$$

$$K_d = 0.5\,L = 4$$

$$\therefore \quad G_c(s) = \frac{0.6\,T}{s}\left(s + \frac{1}{L}\right)^2 = \frac{0.6 \times 94}{s}\left(s + \frac{1}{8}\right)^2$$

$$\therefore \quad G_c(s) = \boxed{\frac{56.4(s+0.125)^2}{s}}$$

Thus, $K_p = 14.1$, $K_i = \frac{1}{T_i} = 0.0625$, $K_d = 4$

Q.17 In an application of the Zielger-Nichols method, a process begins oscillation with a 30 % proportional band in an 11.5 min period. Find the nominal three mode controller settings.

☞ **[SPPU : May-22, Marks 8]**

Ans. : 30 % proportional band will give controller critical gain K_u as,

$$K_u = \frac{100}{PB} = \frac{100}{30} = 3.333$$

$$\therefore \quad K_p = 0.6\,K_u = 0.6 \times 3.333 = 2$$

$$T_i = \text{Integral time} = \frac{T_u}{2}$$

where $T_u = 11.5$ min (given)

$$\therefore \quad T_i = \frac{11.5}{2} = 5.75 \text{ min}$$

$$K_d = \text{Derivative gain} = \frac{T_u}{8} = 1.4375$$

These are nominal three mode controller settings.

6.15 : Concept of Industrial Automation

Q.18 What is industrial automation ? What are its two types ?

Ans. : • In traditional mechanical systems, operated machinery is operated by human intervention.

- Enhancement in technology facilitated the automation of all industrial processing systems, factories, machinery, test facilities, etc.

- In industrial automation use of high control capability devices like PC/PLCs/PACs, etc. are used to increase the efficiency in terms of precision, power and speed of manufacturing or production processes.
- An automation system consists of control systems containing various types of closed-loop control techniques to ensure the process variables follow the set points.
- Some additional functions include functions for computing set points for control systems, plant startup or shutdown, monitoring system performance, equipment scheduling, etc.
- To accomplish automation special dedicated hardware and software products are used.

The two types of industrial automation are,

1. Process plant automation 2. Manufacturing automation

Q.19 Explain the process plant automation system.

Ans. : • In the process industries like pharmaceuticals, petrochemical, cement industry, paper industry, etc. the product is manufactured from various chemical processes on some raw material.

- Fig. Q.19.1 shows process automation system hierarchy.
- As shown in the Fig. Q.19.1 various levels of automation can be explained as below :
  - **Level 0 or plant :** Consists of machines like sensors and actuators for translating the signals from machines to produce control signals.
  - **Level 1 or direct process control :** In this level, information from sensors is used by automatic controllers and monitoring systems.
  - **Level 2 or plant supervisory control :** In this level the targets or set points are set by using automatic controllers.
  - **Level 3 or production scheduling and control :** Various decision-making problems like resource allocation, production target, maintenance management, are handled at this level.

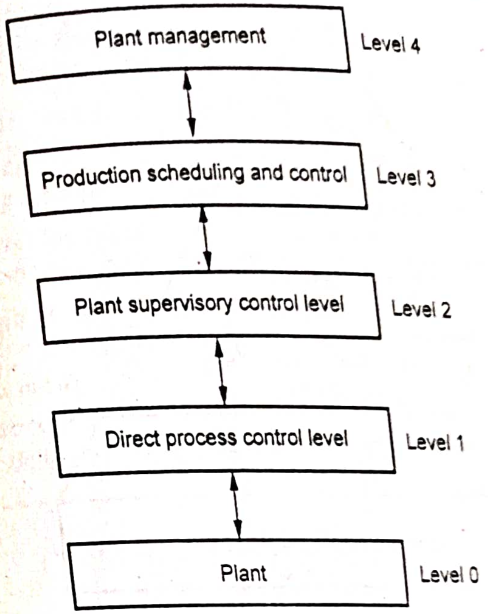


Fig. Q.19.1 Process plant automation system hierarchy

- **Level 4 or plant management :** This is the higher level of the process plant automation. It deals with commercial activities like market analysis, orders and customer analysis, etc. rather than technical activities.

Q.20 Explain the manufacturing automation system.

Ans. : • In manufacturing industries like textile, glass, food, etc. products are made using machines. Automation can be included in various stages of manufacturing like material handling, machining, assembling, inspection and packaging.

- Automation can be made more efficient by means of computer aided control and industrial robot systems.
- Fig. Q.20.1shows the manufacturing automation system hierarchy.
- The functionalities of the levels shown in the Fig. Q.20.1 can be explained as follows :
  - **Machinery level :** Various sensing and actuating devices are used in this level to control the manufacturing process.
  - **Cell or group level :** Operations of a group of machines within manufacturing cells are coordinated at this level.

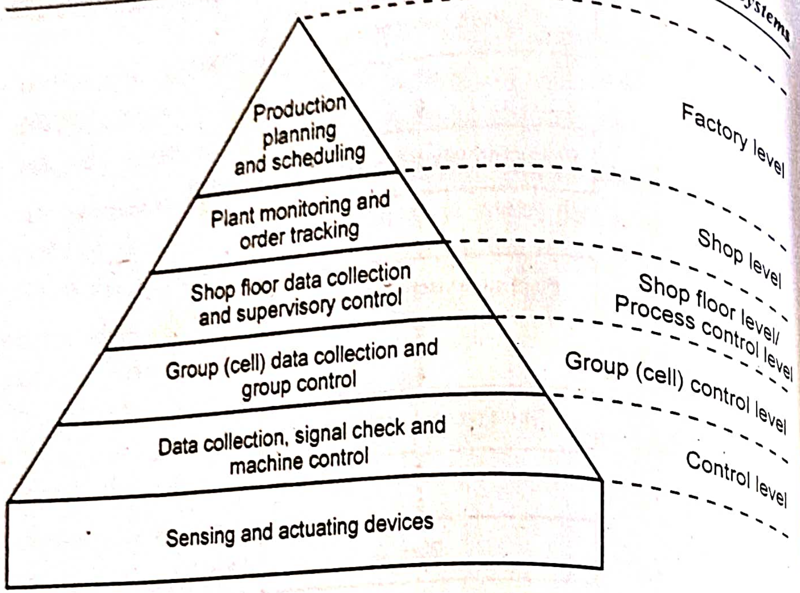


Fig. Q.20.1 Manufacturing automation system hierarchy

- **Shop floor level :** It is a supervisory automated level where supervision and coordination of several manufacturing cells are carried out.
- **Plant level :** Various activities like production monitoring, control, and scheduling, etc. are carried out at this level.
- **Enterprise level :** All the management related activities such as production planning and scheduling, etc. are done at this level.

Q.21 State the advantages of industrial automation.

Ans. : Some of the advantages of industrial automation are :

1. Increased labor productivity
2. Improved product quality
3. Reduced labor or production cost
4. Reduced routine manual tasks
5. Improved safety
6. Assisted remove monitoring.

6.16 : Need of IoT Based Automation

Q.22 Explain the need of IoT based automation.

Ans. : • To understand the concept of Internet of Things (IoT) let's consider a simple scenario. Before some years we were using cellphones to talk and to send messages. Nowadays we are using smartphones which are connected to the internet. We can read a book,watch a movie, listen to songs and can get connected almost to everything apart from just talking and texting, by means of different devices like desktop, tablet, smartphone, etc.

• With this background in simple words the concept of IoT can be explained as taking all the things in the world and connecting them to the internet.

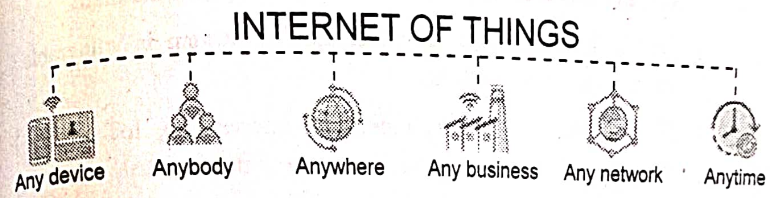


Fig. Q.22.1

- In IoT things can be classified in three categories :
  - Things that collect information and then send it. (For example sensors like temperature sensors, motion sensors, moisture sensors, air quality sensors, light sensors, etc. collect the information and send it to the corresponding device to make it more intelligent)
  - Things that receive information and then act on it. (For example car receives a signal from car keys and the door opens)
  - Things that can do both the tasks mentioned above. (For example the sensors collect information about the soil moisture to tell the farmer how much to water the crops and based irrigation system can automatically turn on as needed, based on how much moisture is in the soil)

Q.23 State the advantages and limitations of IoT.

Ans. : • Some of the benefits of Industrial IoT are :

- High accuracy
- Enhanced efficiency
- Cost-effectiveness
- Low errors
- Lower power needs
- Ease of control
- Quick process completion.

• The limitations of IoT are,

- IoT systems are complex to design, develop and maintain.
- Due to interconnected structure, the IoT system is vulnerable to various network attacks.
- Even If there is no direct personal intervention, IoT system provides substantial personal data in very much detail.
- There is a need for constant updation of the devices.
- Everything will be controlled by networking and Artificial Intelligence resulting in loss of human control at times.

Q.24 Explain the various applications of internet of things.

Ans. :

• Some of the popular IoT applications include :

Application type	Description
Smart home	Smart home makes use of sensor-based devices to facilitate automation. Some of the devices include home appliances, smoke detectors, windows and door locks.
Wearables	Wearable devices are worn by humans on their body. They are smartwatches, smart glasses, etc.

Smart city	A smart city can be developed using advanced information and communication technologies to improve the quality of government services, operational efficiency and share information with the public.
Smart grid	A smart grid is a modern system which delivers electricity. The smart grid enables two-way communication of electricity data unlike traditional grid. It collects real-time data which constitute for electric supply and demand while transmission and distribution process.
Connected car	In smart cars the devices are used for interconnection for various purposes like : Generating a alarm in case of collision, heavy traffic flow and other safety alerts, etc.
Smart retail	Smart retail provides a smart way of shopping, it is built using solutions to convert a conventional physical store into an interactive store. By intelligent systems detailed knowledge of the customers and business, increased sales, etc are obtained to enhance the operation. Example : Paytm has launched its smart retail facility.

END...

Solved Paper

JUNE-2022 (END SEM) [5869] - 247

Course 2019

Time : $2\frac{1}{2}$ Hours] [Max. Marks : 70

Instructions to the candidates :

1) Solve question Q.1 to Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.

2) Figures to the right indicate full marks.

3) Assume the suitable data, if necessary.

Q.1 a) *The characteristics equation of closed loop system is given as*

$1 + G(s)\ H(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$.

Check the stability of system and determine number of closed loop pole lies in RHP of s plane.

(Refer Q.12 of Chapter - 3) [8]

b) *A unity feedback system with open loop transfer function* $G(s) = \frac{K}{(s+1)^4}$. *Plot root locus.*

(Refer Q.24 of Chapter - 3) [10]

OR

Q.2 a) *The characteristics equation of closed loop system is given as* $1 + G(s)\ H(s) = s^3 + 7s^2 + 25s + 39 = 0$. *Determine the number of roots which are lying on left half side of* $\sigma = -1$. **(Refer Q.14 of Chapter - 3)** [8]

b) *Plot a root locus for the system*

$G(s)\ H(s) = \frac{K}{s(s+4)(s^2+4s+13)}\quad 0 < K < \infty$.

(Refer Q.25 of Chapter - 3) [10]

Q.3 a) *Construct Nyquist plot and find phase crossover frequency and gain margin if :* $G(s)\ H(s) = \frac{1}{s(s+1)(s+2)}$. *Also comment on stability.* **(Refer Q.24 of Chapter - 4)** [9]

b) *State the limitations of frequency domain approach.*

(Refer Q.28 of Chapter - 4) [8]

OR

Q.4 a) *Draw Bode plot of the system with open loop transfer function :* $G(s) = \frac{20(s+5)}{s(s+10)}$ *and determine gain margin, phase margin. Also comment on stability.*

(Refer Q.16 of Chapter - 4) [9]

b) *State and explain the various frequency domain specifications.* **(Refer Q.12 of Chapter - 4)** [8]

Q.5 a) *Obtain the controllable and observable canonical state models for the system with transfer function*

$G(s) = \frac{s+3}{s^2+3s+2}$ **(Refer Q.31 of Chapter - 5)** [9]

b) *Define the terms : i) State, ii) State variables, iii) State vector, iv) State space.* **(Refer Q.3 of Chapter - 5)** [9]

OR

Q.6 a) *Find transfer function of* $\dot{X} = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} r(t)$;

$y = [1 \quad 2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. **(Refer Q.13 of Chapter - 5)** [9]

b) *Determine the state transition matrix of state equation* $\dot{X} = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix} x(t)$. **(Refer Q.18 of Chapter - 5)** [9]

Q.7 a) *State the characteristics of P, I and D controllers.* **(Refer Q.3, Q.4 and Q.5 of Chapter - 6)** [9]

b) *What do you understand by integral reset in PID controller ? Explain with suitable example.* **(Refer Q.11 of Chapter - 6)** [8]

OR

Q.8 a) *Describe the Ziegler-Nichols method of process-control loop tuning.* **(Refer Q.12 of Chapter - 6)** [9]

b) *In an application of the Zielger-Nichols method, a process begins oscillation with a 30 % proportional band in 11.5 min period. Find the nominal three mode controller settings.* **(Refer Q.17 of Chapter - 6)** [8]

DECEMBER-2022 (END SEM) [5925] - 217 | Solved Paper

Course 2019

Time : $2\frac{1}{2}$ Hours] **[Max. Marks : 70**

Q.1 a) *Using Routh's and Hurwitz's criteria, comment on the stability if characteristic equation is :*

$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16$.

(Refer Q.12 of Chapter - 3) [8]

b) *Sketch root locus of the unity feedback system with open loop transfer function* $G(s) = \dfrac{K}{s(s+1)(s+4)}$. [10]

Ans. : Refer Q.18 of Chapter - 3 for procedure and nature of root locus and verify breakaway point = – **1.67**, Breakaway point = – **0.46**, Intersection with jω axis = **± j2.**

OR

Q.2 a) *The open loop transfer function of the unity feedback system is* $G(s) = \dfrac{200}{s(s^3 + 6s^2 + 11s + 6)}$. *Using Routh criterion determine stability of the system.*

(Refer similar Q.11 of Chapter - 3) [8]

b) *A unity feedback system has the loop transfer function,* $G(s) = \dfrac{K}{s(s+1)(s+3)(s+4)}$. *Determine : Breakaway points, intersection with imaginary axis. Plot root locus.* [10]

(Refer Q.23 of Chapter - 3)

Q.3 a) *For an unity feedback system with open loop transfer function* $G(s) = \frac{4}{s(s+2)}$. *Determine damping factor, undamped natural frequency, reason peak, resonant frequency.* **[9]**

Ans. : Refer Q.7 of Chapter - 4 for the procedure and verify the answers as : $\xi = 0.5$, $\omega_n = 2$ rad/sec, $M_r = 1.154$, $\omega_r = 1.414$ rad/sec.

b) *Explain Nyquist stability criterion.*

(Refer Q.20 of Chapter - 4) **[8]**

OR

Q.4 a) *If* $G(s)\ H(s) = \frac{1}{s(s+1)}$. *Find resonance peak and resonance frequency.* **[9]**

Ans. : Refer Q.7 of Chapter - 4 for the procedure and verify the answers as : $M_r = 1.154$, $\omega_n = 0.707$ rad/sec.

b) *Explain advantages of frequency domain analysis.*

(Refer Q.27 of Chapter - 4) **[8]**

Q.5 a) *Obtain the expression for state transition matrix using Laplace transform method and state any four properties of state transition matrix.*

(Refer Q.16 and Q.17 of Chapter - 5) **[9]**

b) *Find controllability and observability of the system given by state model.*

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & -2 & 2 \\ 5 & 2 & -8 \end{bmatrix},\ B = \begin{bmatrix} 5 \\ 1 \\ 10 \end{bmatrix},\ C = [10\ \ 15\ \ 11],$$

$D = [0]$ **[9]**

Ans. : Refer Q.26 of Chapter - ... completely controllable and observable.

OR

Q.6 a) *Obtain the state model for the system with transfer function* $\frac{Y(s)}{U(s)} = \frac{3s+4}{s^2+5s+6}$. **(Refer Q.10 of Chapter - 5)** **[9]**

b) *Determine the transition matrix of state equation*

$$X = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} x(t).$$ **[9]**

Ans. : Refer Q.18 of Chapter - 5 for procedure and verify answer as :

$$e^{At} = \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & -1.5e^{-t} + 1.5e^{-3t} \\ 0.5e^{-t} - 0.5e^{-3t} & -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix}$$

Q.7 a) *Explain proportional mode, integral mode and derivative mode.* **(Refer Q.3, Q.4 and Q.5 of Chapter - 6)** **[9]**

b) *What do you mean by industrial automation ? What are its types ? Explain the architecture of an automation.*

(Refer Q.18 and Q.19 of Chapter - 6) **[8]**

OR

Q.8 a) *Explain the Ziegler - Nichols tuning method of tuning a PID controller.* **(Refer Q.12 of Chapter - 6)** **[9]**

b) $\frac{C(s)}{R(s)} = \frac{1}{s^2+s+1}$ *compute the* T_r, T_p, T_s *and %* M_p *for the same. Compare the time domain for proportion gain* $K_p = 20$. **[8]**

Ans. : Comparing denominator with $s^2 + 2\xi\omega_n s + \omega_n^2$

$\omega_n = 1$ rad/sec, $\xi = 0.5$, $\omega_d = \omega_n\sqrt{1-\xi^2} = 0.866$ rad/sec

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = 1.047 \text{ rad}$$

$$\therefore \quad T_r = \frac{\pi-\theta}{\omega_d} = \mathbf{2.42\ sec},\ T_p = \frac{\pi}{\omega_d} = \mathbf{3.63\ sec},$$

$$\therefore \quad T_s = \frac{4}{\xi\omega_n} = \mathbf{8\ sec}$$

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = \mathbf{16.3\ \%}$$

With $K_p = 20$, the system becomes as shown in the Fig. 1.

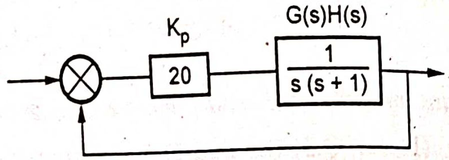


Fig. 1

$\therefore$ $1 + G(s)H(s) = 0$ gives $s^2 + s + 20 = 0$ i.e. $s^2 + 2\xi\omega_n + \omega_n^2 = 0$

$\therefore$ $\omega_n = \sqrt{20} = 4.47,\ \xi = 0.112,\ \omega_d = 4.44,\ \theta = 1.458$ rad

$\therefore$ T_r = **0.379 sec**, T_p = **0.707 sec**, T_s = **7.98 sec**,

$\% M_p$ = **70.18 %**

END...

Notes